VaR and Option Pricing with Normal Mixture Density Functions

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Outline

1. **Are Financial Market Returns Normally Distributed?**
   Test for non-normality
   Evidence of non-normality: FX and Equity Markets

2. **Normal Mixture Densities**
   Definition and Properties of NM Densities
   Parameter Estimation: Real World vs Risk Neutral
   Behavioural Rationale

3. **Application to Value-at-Risk Models**

4. **Application to Option Pricing**

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1. Testing Non-normal Returns

- **Simple statistics:**

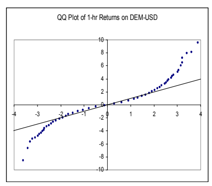
  Skewness $\tau = E\,((X - \mu)/\sigma)^3$

  Excess Kurtosis $\kappa = E\,((X - \mu)/\sigma)^4 - 3$

  where μ and σ are the mean and standard deviation of the returns X

  Approximate standard errors: $\sqrt{(6/n)}$ and $\sqrt{(24/n)}$

- **QQ plots:**



Asymptotic tests:

- e.g. Jarque-Bera

Hedge Fund Returns

	Skewness	Excess Kurtosis
RISK ARBITRAGE		
ZURICH	-3.21	19.77
HENNESSEE	-3.02	18.46
TUNA	-2.25	11.28
ALTVEST	-2.74	14.67
HFR	-3.78	22.53
DISTRESSED		
ZURICH	-2.59	14.05
HENNESSEE	-2.12	9.53
TUNA	-2.02	12.26
ALTVEST	-1.23	4.66
VAN	-0.09	3.52
HFR	-2.18	10.57
EMERGING MARKETS		
ZURICH	-2.41	12.92
HENNESSEE	-0.91	4.70
HFR	-0.81	3.86
CSFB/TREMONT	-0.84	3.09
ALTVEST	-0.70	1.87
VAN	-0.06	1.96

Source: Brooks and Kat (2001)

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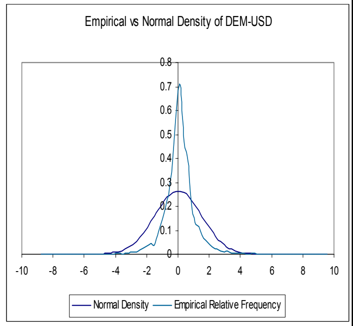
Empirical Evidence: HF Data

Normality statistics for the return on DEM/USD rates

	1-hr	6-hr	12-hr	1-day
Skewness	0.289	0.231	0.198	0.089
XS Kurtosis	8.34	3.43	1.51	0.61
JB	18245	520	55	4.62



Over a time horizon of no more than a few days, there is significant XS kurtosis. This will affect:

- the pricing of short term options
- Intra-day delta hedging of options positions

What is XS Kurtosis?

- XS Kurtosis means that the kurtosis is greater than the normal with with same variance, also termed ‘leptokurtic’
- Leptokurtic is the Latin for ‘thin arch’
- What we mean is that the density has long thin tails: in fact the term ‘fat-tailed’ is a bit of a misnomer
- For example, this trinomial distribution has

  XS kurtosis = (1/2p) – 3

  which is large for *small* p:

  x [w.p. p]

  0 [w.p. 1 – 2p]

  -x [w.p. p]

2. Normal Mixture Densities

- A normal mixture density is a probability weighted sum of normal density functions.
- For example a mixture of two normal densities

  $\phi_1(x) = \phi(x; \mu_1, \sigma_1^2)$ and $\phi_2(x) = \phi(x; \mu_2, \sigma_2^2)$

  is the density:

$$\eta(x) = p\, \phi_1(x) + (1-p)\, \phi_2(x)$$

  That is:

$$p\,[(2\pi\sigma_1^2)^{-1/2} \exp(-((x - \mu_1)/\sigma_1)^2/2)] +$$
$$(1 - p)\,[(2\pi\sigma_2^2)^{-1/2} \exp(-((x - \mu_2)/\sigma_2)^2/2)]$$

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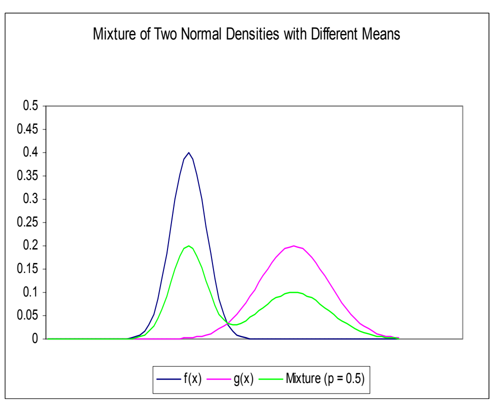
Normal Mixture Distributions



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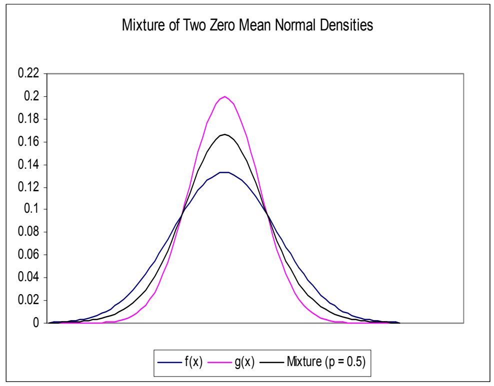
Normal Mixture Densities



Mixing two normal densities with the same mean but different variances gives a symmetric and leptokurtic distribution

Volatility of Normal Mixtures

- If we wish to model only the fat-tails, not the skewness, we can use two normal densities with zero means.
- Assuming means are zero, the variance is just the probability weighted sum of the individual variances
- For example consider a mixture of two zero mean normal densities
  - **one with probability 0.6 and volatility 5%**
  - **the other with probability 0.4 and volatility 14.58%.**
- The mixture density has volatility 10%, since

$$\mathbf{0.6x5^2 + 0.4x14.58^2 = 100}$$

XS Kurtosis of Normal Mixtures

- A zero mean normal mixture with volatility 10% always has fatter tails than a normal density with volatility 10%
- This is because, assuming zero means, the excess kurtosis of a normal mixture $\eta(x) = \Sigma\, p_i\, \phi(x;\, 0,\, \sigma_i^2\,)$ is:

$$\mathbf{3\ [\ (\ \Sigma\ p_i\ \sigma_i^4\ /\ \{\ \Sigma\ p_i\ \sigma_i^2\ \}^2\ )\ -\ 1\ ]}$$

- For example, on the previous slide where:

  $p_1 = 0.6$, $p_2 = 0.4$, $\sigma_1 = 0.05$ and $\sigma_2 = 0.1458$

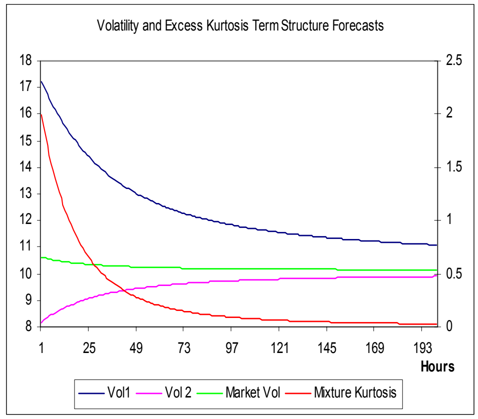
  the XS kurtosis is **2.535**
- But the normal density with the same volatility (10%) has zero XS kurtosis

XS Kurtosis Term Structure

- High frequency returns in liquid markets have significant XS kurtosis and can be modelled by normal mixture densities
- The XS kurtosis disappears when returns are sampled over more than a few days - this is a consequence of the central limit theorem
- There is a behavioural model to support this where traders have heterogeneous expectations of volatility over the *very* short term although their views on volatility may be similar over an horizon of a few days (and then, in all probability, their views about long term volatility will diverge again)

XS Kurtosis Term Structure

- Type j traders have volatility expectations term structure 'vol j' (j = 1,2)
- Using normal mixture model, the aggregate volatility in this example is an almost constant volatility term structure
- However, note the declining XS kurtosis term structure



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Moments of Normal Mixtures

$$\eta(x) = \Sigma\, p_i\, \phi(x;\, \mu_i\, ,\sigma_i^2\,)$$

- Mean $\mu_\eta = \sum p_i \mu_i$
- Variance $\sigma_\eta^{\,2} = \sum p_i \sigma_i^{\,2} + \sum p_i \mu_i^{\,2} - \mu_\eta^{\,2}$
- Skewness $\tau_\eta = \sum p_i [\,(3\mu_i/\sigma_i) + (\mu_i^3/\sigma_i^3)\,]\,(\sigma_i^{\,3}/\,\sigma_\eta^{\,3}) - (3\mu_\eta/\sigma_\eta) - (\mu_\eta^3/\,\sigma_\eta^3)$
- XSKurtosis $\kappa_{e\,\eta} = 3\,\{\,[\,(\sum p_i \sigma_i^{\,4})/(\sum p_i \sigma_i^{\,2})^2\,] - 1\,\}$
(for zero means)

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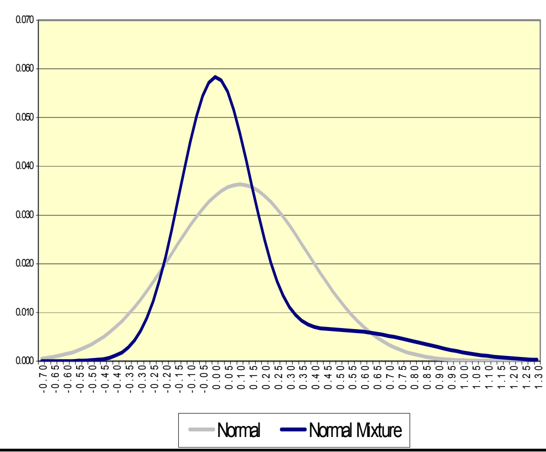
Four Parameter NM $(p, \mu, 0, \sigma_1^2, \sigma_2^2)$

A mixture of two normal densities, one of which has non-zero mean is a four parameter density which is flexible enough to fit most types of returns distributions:

	p	μ	σ_1^2	σ_2^2	τ_η	$\kappa_{e\,\eta}$	
Scenario A	0.8	0.5	0.02	0.1	1.4891	2.5956	$\tau_\eta > 0; \kappa_{e\,\eta} > 0$
Scenario B	0.2	0.5	0.02	0.1	0.1649	-0.5234	$\tau_\eta > 0; \kappa_{e\,\eta} < 0$
Scenario C	0.2	0.5	0.1	0.02	-1.4891	2.5956	$\tau_\eta < 0; \kappa_{e\,\eta} > 0$
Scenario D	0.8	0.5	0.1	0.02	-0.1649	-0.5234	$\tau_\eta < 0; \kappa_{e\,\eta} < 0$

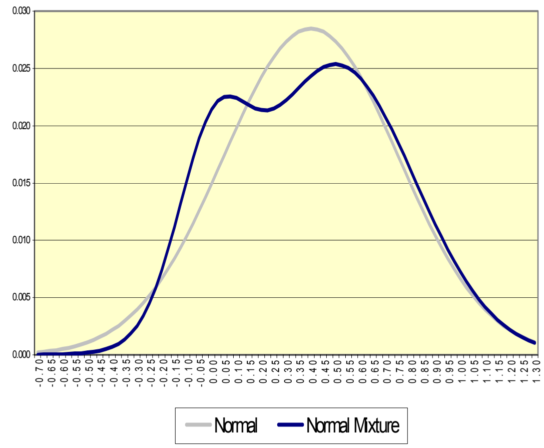
Four Parameter NM $(p, \mu, 0, \sigma_1^2, \sigma_2^2)$

Positive Skewness and Positive XS Kurtosis



Four Parameter NM $(p, \mu, 0, \sigma_1^2, \sigma_2^2)$

Positive Skewness and Negative XS Kurtosis



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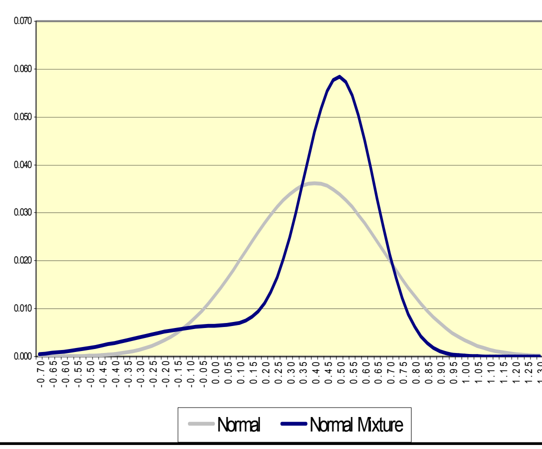
Four Parameter NM $(p, \mu, 0, \sigma_1^2, \sigma_2^2)$

Negative Skewness and Positive XS Kurtosis



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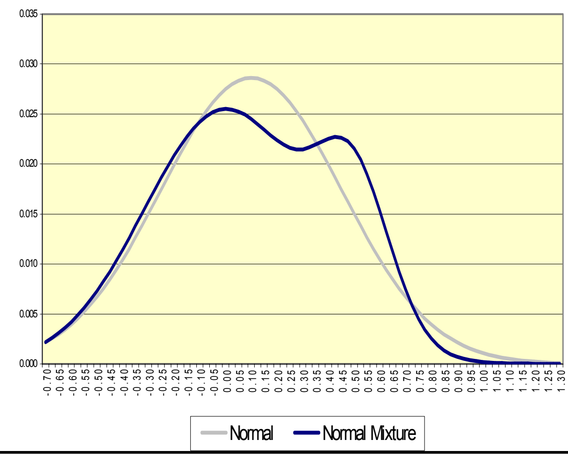
Four Parameter NM $(p, \mu, 0, \sigma_1^2, \sigma_2^2)$

Negative Skewness and Negative XS Kurtosis



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Parameter Estimation

Risk Neutral

– Choose parameters to minimize RMSE between market prices of options, and – assuming we *can* price options under the lognormal mixture process – the model price based on lognormal mixture

– Ritchey (1990), Melick and Thomas (1997)

Real World

– Method of Moments: Choose parameters so that the mean, variance, skewness and excess kurtosis under the normal mixture is the same is that observed on empirical returns.

– Hull and White (1998), Alexander and Naranathan (2001)

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3. Application to VaR Models

Background:

How to obtain VaR estimates: simulation vs analytic

A comparison of methodologies:

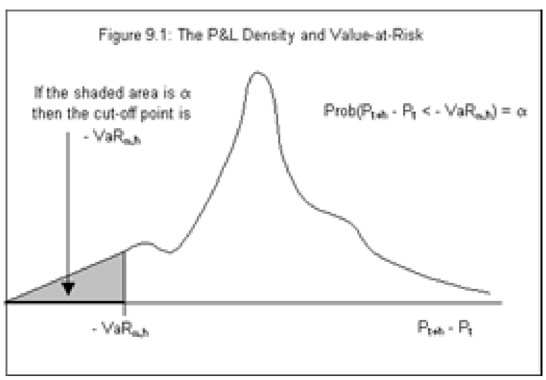
- RiskMetrics
- Orthogonal Methods

Non-normal VaR:

Why use non-normal VaR estimates?

Normal mixture VaR estimates

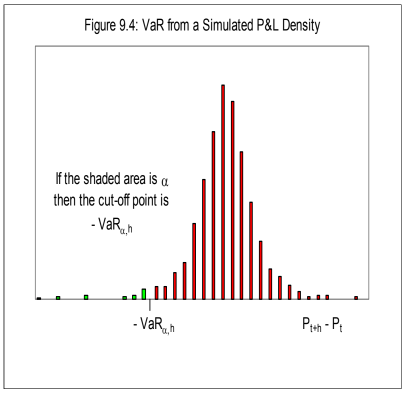
3.1. Background



Source: Market Models (C.Alexander, Wiley 2001)

Value-at-Risk (VaR) is the anticipated loss from an adverse market movement with a specified probability over a particular period of time

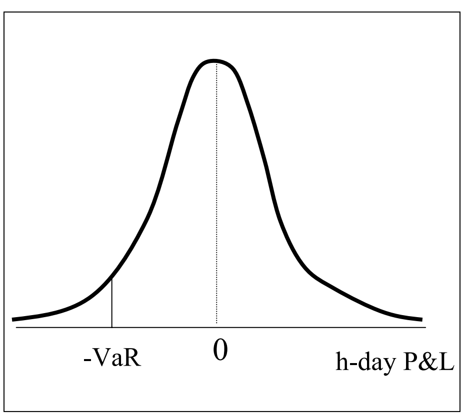
Historical Simulation



Does not assume normality

- To obtain robust estimates one has to use many years of good daily data, which may not be easy to obtain
- Only one estimate is obtained, which is a 'hybrid' of VaR in normal market circumstances and VaR in extreme market circumstances

Covariance (RiskMetrics) VaR Model



- Assumes P&L is normally distributed
- Analytic formula for VaR:

VaR = $Z_\alpha \sigma$

Z_α is the critical value of N(0,1)

σ is the volatility of P&L, given by

$\sqrt{p'Vp}$

Important Note: most analytic VaR models are not good for long-only positions

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Covariance Matrix, V

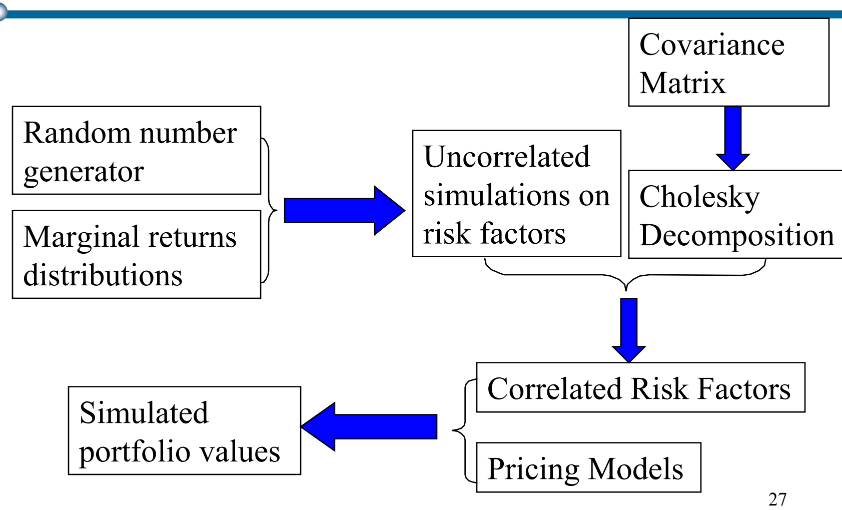
The covariance matrix of asset or risk factor h-day returns is central to both the ‘Riskmetrics” and the MC VaR models:

$$\begin{pmatrix} V(X_1) & COV(X_1,X_2) & \dots & \dots & COV(X_1,X_k) \\ COV(X_1,X_2) & V(X_2) & \dots & \dots & COV(X_2,X_k) \\ COV(X_1,X_3) & COV(X_2,X_3) & V(X_3) & \dots & COV(X_3,X_k) \\ \dots & \dots & \dots & \dots & \dots \\ COV(X_1,X_k) & \dots & \dots & \dots & V(X_k) \end{pmatrix}$$

Monte Carlo VaR

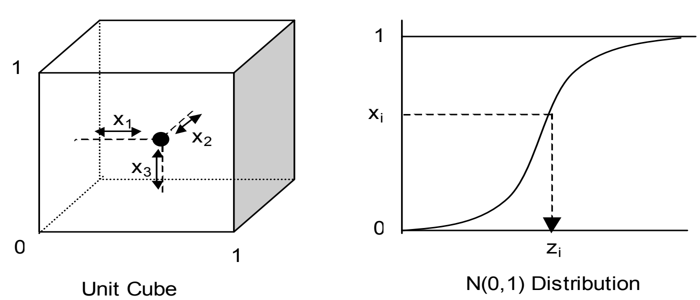
- Monte Carlo VaR is similar to historical VaR in that a set of scenarios on the risk factors are put into the option pricing functions to get a set of portfolio values, and then these portfolio values are used to generate a P&L distribution.
- However the scenarios are forward looking, over a pre determined risk horizon of h days.
- Correlated scenarios are used, and these are generated using an h day covariance matrix of risk factor returns (only historical simulation does not use the covariance matrix)

Monte Carlo Simulation

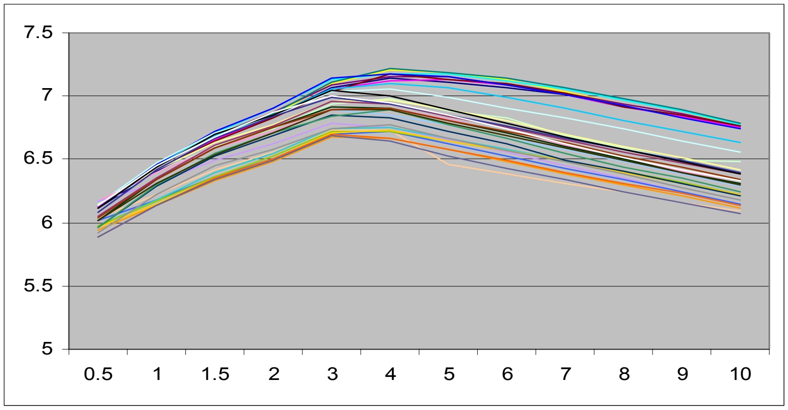


Future Simulations

Figure 9.5: Sampling the Hypercube and Simulating Independent N(0,1) Observations



Correlated Simulations



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Estimation of Covariance Matrix

- Many different methodologies available
  - 'Historical'
  - Exponential weighted moving averages (EWMA)
  - Generalized autoregressive conditional heteroscedasticity (GARCH)
- Difficult to assess which is most accurate
  - Unlike prices, volatilities and correlations are not observable in the market

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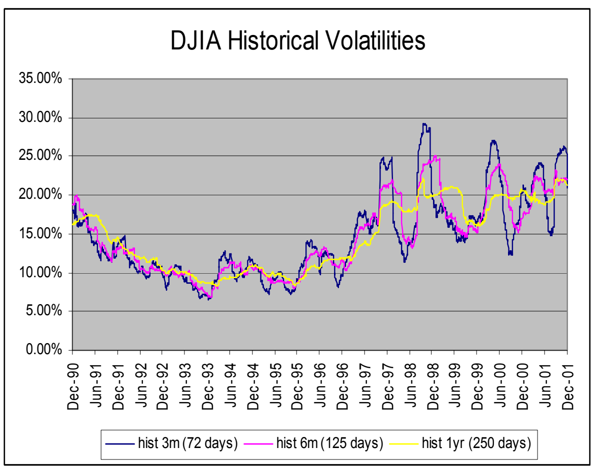
RiskMetrics Data

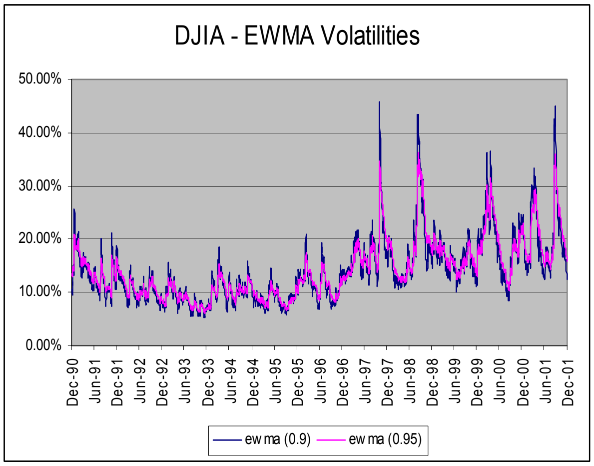
- Very large covariance matrices of the returns to many risk factors: major foreign exchange rates, money market rates, equity indices, interest rates and some key commodities.
- Downloadable from the internet: www.riskmetrics.com.
- There are three types of covariance matrix:
  - **1-day matrix** (EWMA with $\lambda = 0.94$)
  - **1-month (25-day) matrix** (EWMA on daily returns with $\lambda = 0.97$ and then the matrix is multiplied by 25)
  - **a 'regulatory' matrix** (Historical with n = 250)
- More details are in the *RiskMetrics Technical Document* (J.P. Morgan and Reuters, 1996).

Problems with RiskMetrics



A major problem with the 'historical' estimates is that extreme events are just as important to current estimates, whether they occurred yesterday or at any other time during the historical period.

Problems with RiskMetrics



In the EWMA matrices, the same value of the exponential smoothing constant λ must be used for all markets. This is necessary, otherwise the covariance matrix would not be positive semi-definite

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Which Estimate?

Date	3m	6m	1yr	ewma(0.9)	ewma(0.95)
31-Dec-01	19.02%	22.20%	21.35%	12.41%	15.85%

- These are all estimates of the same thing – the constant volatility in the GBM price process
- There is no ‘optimal’ way to choose the best parameter values – which of these estimates you decide use is just a matter of subjective choice.

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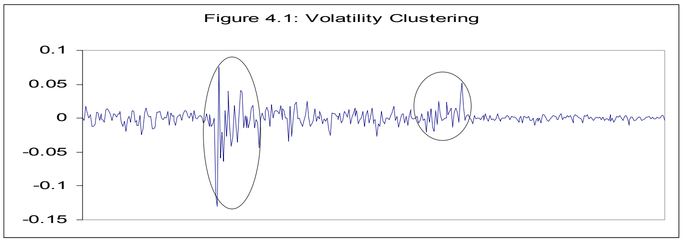
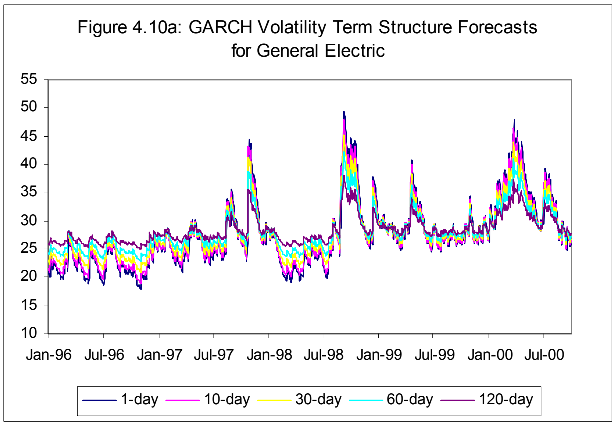
Problems with RiskMetrics

- **Constant volatility:**
  - 'Square root of time' rule:
    h-day covariance matrix = h * 1-day covariance matrix
  - Current estimate is the forecast for all risk horizons
  - Assumption is that returns are normal and independent
- **But returns are not normal, or independent.**



The Advantage of GARCH Models



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The Problem with GARCH Models

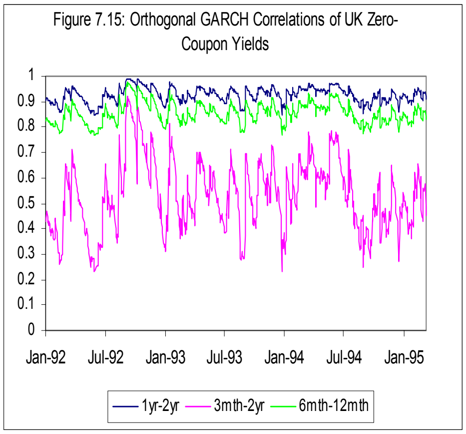
- Computational problems are inevitable if one attempts direct estimation of full multivariate GARCH models for large-dimensional systems.
- Even the simplest parameterization of a bivariate GARCH(1,1) model, the *diagonal vech,* has eleven parameters:

$$\sigma^2_{1,t} = \omega_1 + \alpha_1 \varepsilon^2_{1,t-1} + \beta_1 \sigma^2_{1,t-1}$$
$$\sigma^2_{2,t} = \omega_2 + \alpha_2 \varepsilon^2_{2,t-1} + \beta_2 \sigma^2_{2,t-1}$$
$$\sigma_{12,t} = \omega_3 + \alpha_3 \varepsilon_{1,t-1}\varepsilon_{2,t-1} + \beta_3 \sigma_{12,t-1}$$

Orthogonal Methods

- Several papers (www.ismacentre.rdg.ac.uk) explain how to use principal component analysis to generate large dimensional covariance matrices using EWMA or GARCH:
    - **O-EWMA**
    - **O-GARCH**
- This method avoids the problems with the RiskMetrics data that were just outlined:
    - Positive definiteness can be assured without using the same smoothing constant for all markets;
    - It conforms to regulators requirements on historic data
- However the O-EWMA model is still based on constant volatility and correlation assumptions (≈ ‘square root of time’ rule)

Why Use O-GARCH?



- Gives convergent term structures
- Simple to compute (only univariate GARCH volatility models)
- Positive semi-definiteness assured
- Produces more robust measures of risk
- Copes with missing data
- Copes with illiquid markets

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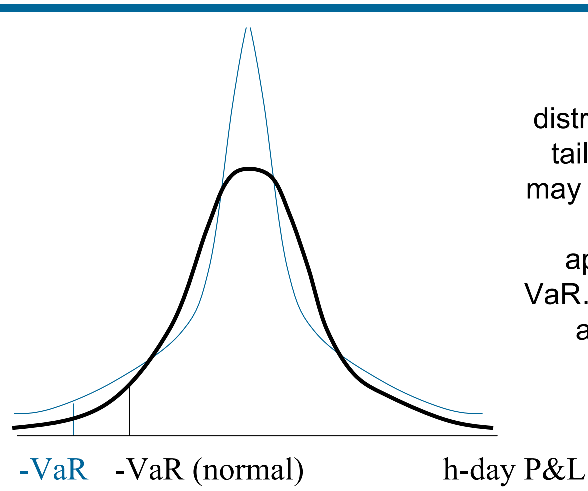
3.2 Non-Normal VaR



If the P&L distribution has fat-tails, actual VaR may be greater than the normal approximation VaR…….but it could also be less!

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Scenarios for Equity VaR

- Normal mixtures are useful to model scenarios in markets with jumps
- For example, the long term average equity index volatility may be 20%, but in your scenario there may be a 1% chance of a market crash, where volatility would be 80%
- This can be modelled by a mixture of two zero mean normal densities with

p = 0.99, vol_1 =20% and vol_2 = 80%

- How should we estimate the equity VaR from taking a long position on this index?

Normal Mixture VaR

- Recall that the normal covariance VaR_α estimate $\approx Z_\alpha\sigma$ where Z_α is the critical value of a standard normal variate and σ is the P&L volatility.
- In this model the P&L variations are assumed to be normal, and so a simple analytic formula for VaR could be derived.
- If we now assume that P&L variations have normal mixture distributions there is no analytic formula for VaR. However it is a simple matter to use a numerical algorithm to obtain a normal mixture covariance VaR measure.
- If we know the probabilities and the volatilities of the normal mixture, we can 'back-out' the VaR_α estimate:

$$\text{prob}(P\&L < -VaR_\alpha) = \Sigma\, p_i \,\text{prob}\,(Z < (-VaR_\alpha / \sigma_i)) = \alpha$$

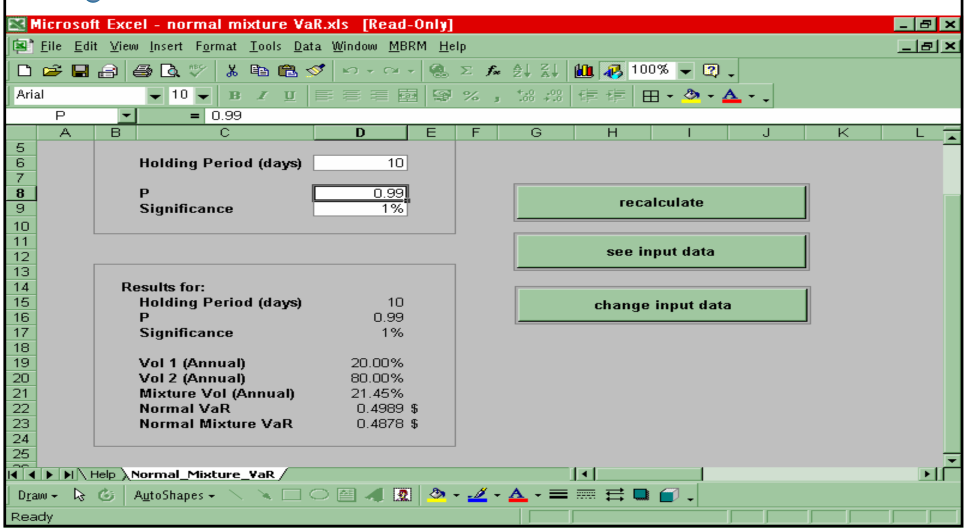
Example: 10-day 1% VaR

- $Vol_1 = 20\% \Rightarrow$ 10-day standard deviation = $0.2/\sqrt{250/10} = 0.2/5 = 0.04$
- $Vol_2 = 80\% \Rightarrow$ 10-day standard deviation = $0.8/\sqrt{250/10} = 0.8/5 = 0.16$
- So 1% 10-day VaR is found by solving:

0.99 Prob(Z < VaR/0.04) + 0.01 Prob(Z < VaR/0.16) = 0.01

- This gives a 10-day 1% VaR = 0.4878$ for each $ invested
- Notice that this is *less* than 0.4989$, the 1% VaR based on the normality assumption with an equivalent volatility (21.45%)
- However, the 0.1% VaR is greater under the normal mixture: it is 1.025$ for each $ invested, but the normal VaR is only 0.66$ per $ invested.

Example



Scenarios for Diversified Portfolios

- **Portfolio diversification**: risk reduction achieved by
  - Long/long (or short/short) on negatively correlated pairs
  - Long/short positions on highly correlated pairs
- **Stress Covariance Matrix**: Long/long or short/short equity

$$\begin{pmatrix} 0.04 & 0.04 & \dots & \dots & 0.04 \\ 0.04 & 0.04 & \dots & \dots & 0.04 \\ 0.04 & 0.04 & 0.04 & \dots & 0.04 \\ \dots & \dots & \dots & \dots & \dots \\ 0.04 & \dots & \dots & \dots & 0.04 \end{pmatrix}$$

*NB. 10-day variance 0.04 ⇒ volatility of √0.04*5 = 100%*
Cov = 0.04 ⇒ correlation = 1

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Scenarios for Diversified Portfolios

- **Stress Covariance Matrix**: Long/short equity:

$$\begin{pmatrix} 0.04 & 0 & \dots & \dots & 0 \\ 0 & 0.04 & \dots & \dots & 0 \\ 0 & 0 & 0.04 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0.04 \end{pmatrix}$$

OR

$$\begin{pmatrix} 0.04 & -0.04 & \dots & \dots & -0.04 \\ -0.04 & 0.04 & \dots & \dots & -0.04 \\ -0.04 & -0.04 & 0.04 & \dots & -0.04 \\ \dots & \dots & \dots & \dots & \dots \\ -0.04 & \dots & \dots & \dots & 0.04 \end{pmatrix}$$

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Example

Microsoft Excel - normal mixture VaR.xls [Read-Only]

Position ($)
1
1
-1

Matrix V1

	Asset1	Asset2	Asset3
Asset1	0.0036	0	0.0036
Asset2	0	0.0036	0.0016
Asset3	0.0036	0.0016	0.0036

Matrix V2

	Asset1	Asset2	Asset3
Asset1	0.04	0.04	0
Asset2	0.04	0.04	0
Asset3	0	0	0.04

Normal market (V1): 30% volatility; positive correlation (asset1, asset 3) and (asset2, asset3); zero correlation: (asset1, asset2).

Stress market (V2): correlation structure breaks down and volatilities increase to 100%

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Example (Cont'd)

1% normal mixture VaR = 2.87$

But normal VaR seriously under estimates:

1% normal VaR = 1.66$

Microsoft Excel - normal mixture VaR.xls [Read-Only]

Results for:		
Holding Period (days)	10	
P	0.9	
Significance	1%	
Vol 1 (Annual)	10.00%	
Vol 2 (Annual)	223.61%	
Mixture Vol (Annual)	71.34%	
Normal VaR	1.6597	$
Normal Mixture VaR	2.8656	$

Summary: Normal Mixture VaR

- Standard 'normal' VaR estimates are inadequate for portfolio risk management
- Even when they use the best possible betas and covariance matrices, there is still the problem that portfolio returns are often highly non-normal
- Even in the exception, where portfolio returns may be fairly normal, they will not normal under stressed market circumstances
- The normal mixture model presented here has useful applications to calculating
  - Market Risk Capital Requirements when market returns are not normal, and
  - Scenario VaR that incorporates the probability that the market will crash

4. Option Pricing with Normal Mixture Densities

- Recent research by Brigo and Mercurio (2001) – see www.fabiomercurio.it
- Option may be priced under the assumption of normal mixture price processes
- There is no additional source of uncertainty in the model – it is a local volatility model and not a stochastic volatility
- Simple formulae for pricing European options with the assumption of NM price process
- Explanation for the smile!

Lognormal Mixture Densities

- Log returns (X) generated by normal mixture
  $\Rightarrow$ prices (S) generated by lognormal mixture:
- For example a mixture of two lognormal densities
  $f_1(s) = f(s; \mu_1, \sigma_1^2)$ and $f_2(s) = f(s; \mu_2, \sigma_2^2)$
  is the density:

$$\mathbf{g(s) = p\, f_1(s) + (1\text{-}p)\, f_2(s)}$$

That is:

$$p\,[(2\pi\sigma_1^2 s^2)^{-1/2} \exp(-((\mathit{ln}(s) - \mu_1)/\sigma_1)^2/2)] +$$
$$(1 - p)\,[(2\pi\sigma_2^2 s^2)^{-1/2} \exp(-((\mathit{ln}(s) - \mu_2)/\sigma_2)^2/2)]$$

Lognormal Mixture Price Process

- Suppose that the risk-neutral density at time t of the asset prices is a lognormal mixture density

$$g_t(s) = p_1\, f_{1,t}(s) + \ldots + p_n\, f_{n,t}(s)$$

  where at each time t, $f_{i,t}(s)$ has mean μ and variance $\sigma_{i,t}^2$
- Then Brigo and Mercurio (2001) showed that the underlying asset's price process is a lognormal mixture process

$$\mathbf{dS/S = \mu\, dt + \sigma(S, t)\, dW}$$

  where the local volatility $\sigma(S, t)$ is given by

$$\boldsymbol{\sigma(S, t)^2 = \Sigma\, p_{i,t}^{*}\, \sigma_{i,t}^2}$$

  and $p^*_{I,t} = p_i\,[f_{i,t}(s) / g_t(s)]$ so $\Sigma\, p_i^* = 1$.

Normal Mixture Option Prices

- Notice that with the lognormal mixture price process the market is still complete so we can price options without introducing a subjective risk premium
- Denote by $F(\sigma)$ the Black-Scholes price for a simple European option
- Suppose returns are generated by a mixture of normal distributions with variances σ_i^2 and probabilities p_i
- Then

Normal mixture option price = $\Sigma\, p_i\, F(\sigma_i)$

Calibration to Historical Data

Method of Moments

- **Historical data on daily returns:**

  Daily XS kurtosis = 2.53

  Annual volatility 20%
- **Mixture of two zero mean normal densities**

  p = 0.6 annual volatility = 10% (variance = 0.01)

  (1- p) = 0.4 annual volatility = 29.15% (variance = 0.085)

  Annual Variance = $p\sigma_1^2 + (1\text{-}p)\,\sigma_2^2 = 0.04$

  Daily XS kurtosis = 2.53 (using formula on slide 12)
- With these parameters the normal mixture option prices are

0.6 F(10%) + 0.4 F(29.15%)

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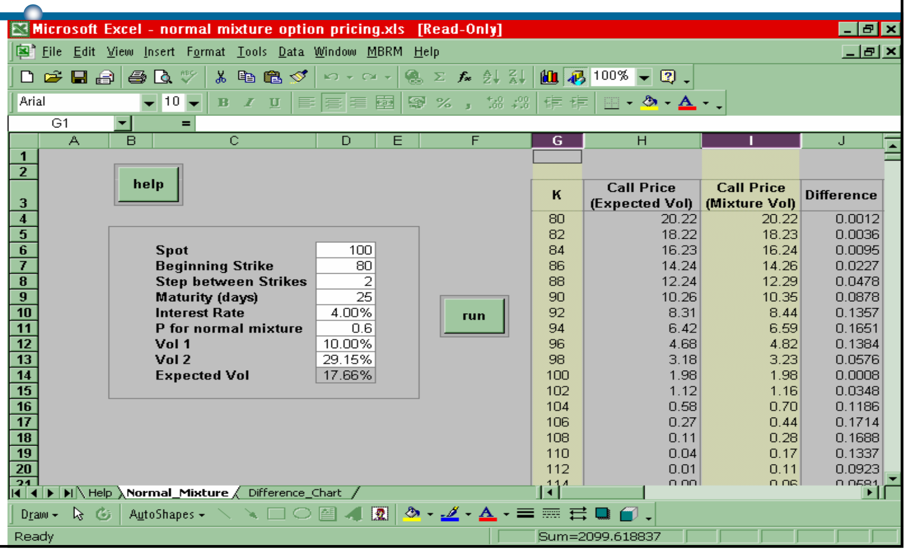
Example



Microsoft Excel - normal mixture option pricing.xls [Read-Only]

Spot	100
Beginning Strike	80
Step between Strikes	2
Maturity (days)	25
Interest Rate	4.00%
P for normal mixture	0.6
Vol 1	10.00%
Vol 2	29.15%
Expected Vol	17.66%

K	Call Price (Expected Vol)	Call Price (Mixture Vol)	Difference
80	20.22	20.22	0.0012
82	18.22	18.23	0.0036
84	16.23	16.24	0.0095
86	14.24	14.26	0.0227
88	12.24	12.29	0.0478
90	10.26	10.35	0.0878
92	8.31	8.44	0.1357
94	6.42	6.59	0.1651
96	4.68	4.82	0.1384
98	3.18	3.23	0.0576
100	1.98	1.98	0.0008
102	1.12	1.16	0.0348
104	0.58	0.70	0.1186
106	0.27	0.44	0.1714
108	0.11	0.28	0.1688
110	0.04	0.17	0.1337
112	0.01	0.11	0.0923

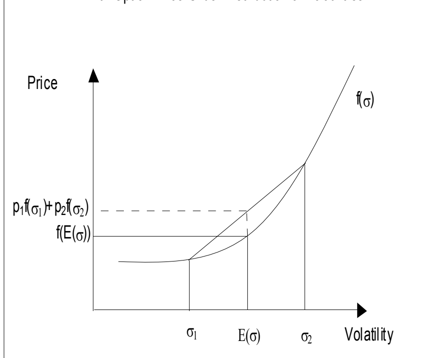
Comparison with BS Prices

- Several studies (e.g. Hull and White, 1987) that use historical data to obtain option prices based on non-normal price processes, and then compare these prices with BS option prices, conclude that BS prices are too low for OTM and ITM options and *too high* for ATM options
- This is an erroneous conclusion because they compare the non-normal option price with the BS price based on the same expected **variance**
- In fact, the non-normal price should be compared with the BS price that has the same expected **volatility** - since $E(\sqrt{X}) \neq \sqrt{E(X)}$, the expected volatility is NOT the square root of the expected variance!

Comparison with BS Prices

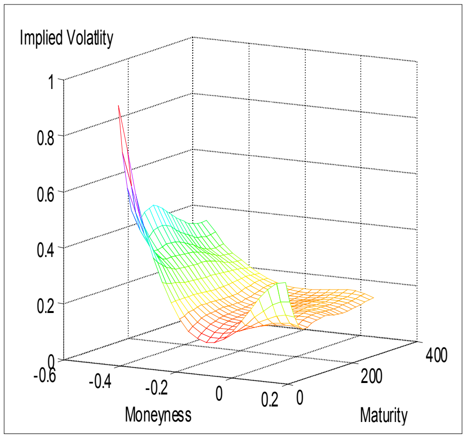
- ATM options are linear in volatility therefore the BS price of an ATM option that is based on the expected volatility, will be the same as the normal mixture option price.
- However simple OTM and ITM are convex in volatility so the normal mixture price will be **greater** than the BS price.

Figure 10.9: Option Price Under Expected Volatility Compared with Option Price Under Distribution of Volatilities



Explanation for the Smile

- One reason for the smile that is observed in implied volatilities is that returns are leptokurtic
- Thus OTM and ITM options are under-priced by BS, and their implied volatilities must be greater than ATM implied
- However NM option prices for OTM and ITM options are greater than BS prices, so we have explained at least part of the smile.



Comparison with BS Prices

K	t	Call Price (BS)	Call Price (Mixture)	Difference
90.00	0.08	10.055	10.107	-0.052
92.00	0.08	8.148	8.215	-0.067
94.00	0.08	6.347	6.412	-0.065
96.00	0.08	4.714	4.757	-0.043
98.00	0.08	3.314	3.328	-0.013
100.00	0.08	2.191	2.191	0.000
102.00	0.08	1.355	1.368	-0.013
104.00	0.08	0.781	0.823	-0.042
106.00	0.08	0.419	0.485	-0.066
108.00	0.08	0.209	0.283	-0.074
110.00	0.08	0.097	0.163	-0.066

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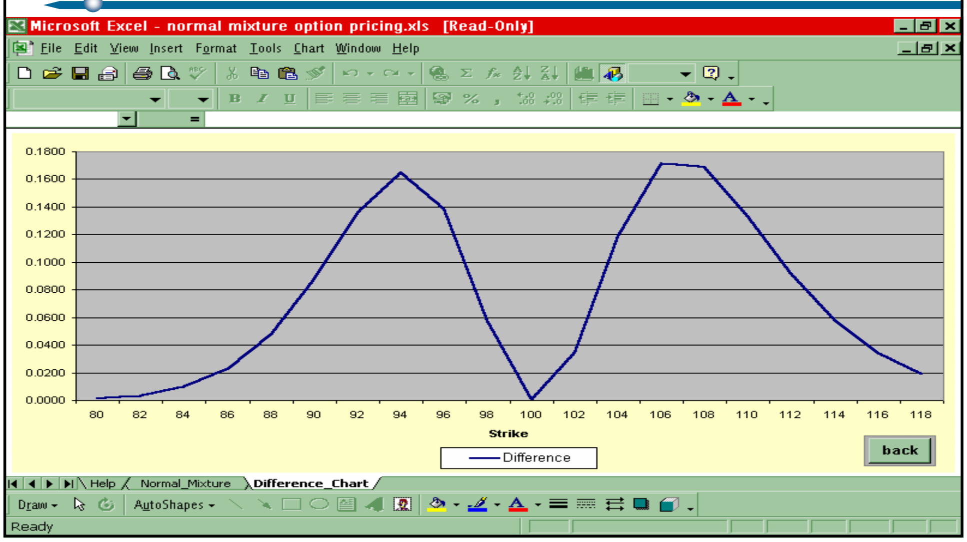
NM Price – BS price



New greek: $\partial^2 f/\partial\sigma^2$

NM Price – BS price $\propto \partial^2 f/\partial\sigma^2$

- To see this, take Taylor expansion about $E(\sigma)$ and then take expectations:

$$f(\sigma) \approx f(E(\sigma)) + (\partial f/\partial\sigma)(\sigma - E(\sigma)) + \tfrac{1}{2}(\partial^2 f/\partial\sigma^2)(\sigma - E(\sigma))^2$$

$$E(f(\sigma)) \approx f(E(\sigma)) + \tfrac{1}{2}(\partial^2 f/\partial\sigma^2)\, V(\sigma)$$

$$\textit{NM Price} - \textit{BS price} = E(f(\sigma)) - f(E(\sigma)) \approx \tfrac{1}{2}(\partial^2 f/\partial\sigma^2)\, V(\sigma)$$

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Relation with the Smile

$$E(f(\sigma)) \approx f(E(\sigma)) + \tfrac{1}{2}(\partial^2 f/\partial\sigma^2)\, V(\sigma)$$

- But

$$f(E(\sigma) + \Delta\sigma) \approx f(E(\sigma)) + \partial f/\partial\sigma\, \Delta\sigma$$

- So

$$\Delta\sigma \approx \tfrac{1}{2}(\partial^2 f/\partial\sigma^2)\, V(\sigma) / \partial f/\partial\sigma$$

- And it can be shown that for European options:

$$\partial^2 f/\partial\sigma^2 / \partial f/\partial\sigma = \sigma^{-1} + A\sigma^{-2}$$

  where A = xy (these are the x and y in the BS formula)
- Interpret $\Delta\sigma$ as the extra implied volatility in the smile
- The smile always increases, despite the 'M'-shape of $\partial^2 f/\partial\sigma^2$

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Summary

- Option prices that include uncertainty in volatility – such as the normal mixture (NM) option pricing model of this lecture, should be compared with normal Black-Scholes (BS) option prices with the same expected volatility
- Earlier work has made the mistake of comparing non-normal prices with the BS prices having the same expected variance – but this leads to the erroneous conclusion that the BS model under-prices ATM options by a long way.
- Normal mixture option prices are the same as BS prices of ATM options but, for OTM options, the NM prices are greater than the BS prices.
- Hence the NM model explains at least part of the smile.