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Robust Monetary Policy

DISSERTATION PAPER

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1. Introduction

According to Alan Greenspan (2003), “Uncertainty is not just an important feature of the monetary policy landscape; it is the defining characteristic of that landscape”. In fact, the recognition that all monetary policymakers must bow to the presence of uncertainty appears to underlie Greenspan’s (2003) view that central banks are driven to a “risk management” approach to policy, whereby policymakers “need to reach a judgement about the probabilities, costs, and the benefits of the various possible outcomes under alternative choices for policy”.

Uncertainty comes in many forms. One obvious form is simply ignorance about the shocks that will disturb the economy in the future (oil prices, for example). Other forms of uncertainty, perhaps more insidious can also have resounding implications on how policy should be conducted, three of which are data uncertainty, parameter uncertainty, and model uncertainty.

Data uncertainty

One form of uncertainty that is ever present is data uncertainty. Consider the US economy’s real GDP. For each and every quarter of the year, three estimates of real GDP are released: an advance estimate, a preliminary estimate, and a final estimate. As successive estimates are released, a greater fraction of the estimate is actually measured and less is imputed. But some imputation is involved even for the final GDP released. In fact, the final GDP estimate is not final. Every year a benchmark revision occurs in which previous estimates of real GDP are revised, going back several years. Try as we might, due to measurement difficulties of one sort or another, we can never know what the economy’s real GDP actually is, or was. This is data uncertainty.

Parameter uncertainty

Distinct from data uncertainty is parameter uncertainty. Economists use models to understand how the economy might respond when stimulated in certain ways, and to create forecasts. These economic models contain parameters that govern the interactions that occur within the model, such as how sensitive consumption or investment is to a 1 percentage point change in the real interest rate. While economists can use statistical techniques to try to estimate these parameters, ultimately their value remains very much uncertain quantities

Model uncertainty and model averaging

While there is uncertainty about the data that enter into economic models and about the parameters that govern economic models, the fact that economists often approach macroeconomic data armed with different models of the economy suggests that uncertainty, or ambiguity, about the model could also be potentially important. From a policymaking perspective, it is quite possible, indeed reasonable, to think that policymakers may have several models at their disposal, perhaps reflecting competing economic theories, each of which could justifiably be viewed as a reasonable approximation of the interrelationships at work in the actual economy.

A policy can be made "robust" to model uncertainty by designing it to perform well on average across all of the available fully specified models rather than to reign supreme in any particular model (McCallum 1988). This model-averaging approach is taken in Levin, Wieland, and Williams (2003), who use five disparate macroeconometric models of the U. S. economy to study how best to conduct monetary policy when facing model uncertainty. Focusing on simple rules in which the Federal Reserve is assumed to set the federal funds rate in response to inflation, the output gap (that is, the difference between actual output and an estimate of potential output), and the lagged federal funds rate, they identify a particular policy rule that is able to perform well across all five models. The policy rule that they identify is one that contains a short-term forecast of future inflation, incorporates a large response to the output gap, and that involves considerable "gradualism," or interest rate smoothing.

Although the model averaging approach allows us to get a handle on how to think about model uncertainty at the level of the policymaker, it is less clear what the approach has to say about the views of the households and firms that make up the economy.

Model uncertainty and robust control

The model-averaging approach to model uncertainty is not possible when policymakers cannot articulate and specify the various models that they wish to be robust against and therefore cannot assign probabilities to each of the models. This situation is known as Knightian uncertainty (Knight 1921). In such environments, the robust control approach comes into play. Robust control suggests that policymakers should formulate policy to guard against the worst form of model misspecification that is possible. Thus, rather than focusing on the “most likely” outcome or on the average outcome, robust control argues that policymakers should focus on and defends against the worst-case outcome.

While the robust control approach may suggest some paranoia on the part of the policymaker, the intuition for robust control can be found in such common expressions as “expect the unexpected” and “hope for the best, but prepare for the worst.”A valuable feature of the robust control approach is that it allows us to think about and combine model misspecification from the perspective of the policymaker with model misspecification from the perspective of households and firms. After all, there is no reason to think that policymakers are the only people who have to worry about model misspecification.

The theory establishing that robust control methods can be applied to economic problems has been developed largely in a series of contributions by Hansen and Sargent, contributions that are well summarized in Hansen and Sargent (2006). Among other things, Hansen and Sargent show how to set up and solve discounted robust control problems, and they develop methods to solve for robust policies in backward-looking models and in forward-looking models with commitment. Giordani and Soderlind (2004) extend these methods to forward-looking models with discretion and to simple rules.

A critical component in the application of robust control is the reference model. A reference model is a structural model, possibly arrived at through some (nonmodeled) learning process that is thought to be a good approximation to the underlying datagenerating process. The methods described in Hansen and Sargent (2006) and Giordani and Soderlind (2004) require that this reference model be written in a state-space form, following the literature on traditional (nonrobust) optimal control. As discussed in Dennis (2006), while state-space methods allow models to be expressed in a form that contains only first-order dynamics, they also have drawbacks. In particular, many models cannot be expressed easily in a state-space form, especially medium- to largescale models for which the necessary manipulations are often prohibitive. For robust control problems, the state-space formulation has an additional important implication in that the policymaker and the fictitious “evil agent” are not treated symmetrically. Specifically, the planner’s decisions can affect current period outcomes both directly and through private sector expectations, while the evil agent’s decisions can only affect current period outcomes through private sector expectations. As we show in this paper, this feature of the traditional robust control setup means that the evil agent will introduce specification errors by changing the conditional means of the shock processes, but not their conditional volatility.

An alternative set of tools to solve robust control problems is based on the solution methods developed by Dennis (2006) that have the advantage that they do not require the reference model to be written in a state-space form. Instead they allow the reference model to be written in structural form, which is more flexible and generally much easier to attain than is a state-space form. The structural form also allows us to treat the policymaker and the evil agent symmetrically, giving rise to the result that the evil agent will optimally choose to change the conditional volatility of the shocks in addition to their conditional means.

2. The model

When solving robust control problems there are generally two distinct equilibria that are of interest. The first is the "worst-case" equilibrium, which is the equilibrium that pertains when the policymaker and private agents design policy and form expectations based on the worst-case misspecification and the worst-case misspecification is realized. The second is the "approximating" equilibrium, which is the equilibrium that pertains when the policymaker and private agents design policy and form expectations based on the worst-case misspecification, but the reference model transpires to be specified correctly. In this section we outline how state-space methods can be used to obtain these two equilibria, setting the scene for the structural-form analysis that follows.

According to the state – space formulation, the economic environment is one in which the behavior of an $n \times 1$ vector of endogenous variables, z_t, consisting of n_1 predetermined variables, z_{1t}, and $n_2 \, (n_2 = n - n_1)$ non predetermined variables, z_{2t}, are governed by the reference model

$$z_{1t+1} = A_{11} z_{1t} + A_{12} z_{2t} + B_1 u_t + C_1 \varepsilon_{1t+1}, \tag{1}$$

$$E_t z_{2t+1} = A_{21} z_{1t} + A_{22} z_{2t} + B_2 u_t, \tag{2}$$

where u_t is a $p \times 1$ vector of control variables, $\varepsilon_{1t} \approx iid[0, I_s]$ is an $s \times 1$ vector, $s \leq n_1$ vector of white – noise innovations, and E_t is the mathematical expectations operator conditional upon information available up to and including period t. The reference model is the model that private agents and the policy maker believe most accurately describes the data generating process. The matrices $A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2$ contain structural parameteres and are conformable with z_{1t}, z_{2t} and u_t as necessary. The matrix C_1 is determined to insure that ε_{1t} has the identity matrix as its variance – covariance matrix.

The policymaker's problem is to choose a sequence for its control variables, $\{u_t\}_0^\infty$, to minimize the objective function

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[z_t' R z_t + 2 z_t' U u_t + u_t' Q u_t \right], \tag{3}$$

where $\beta \in (0,1)$ is the discount factor. The weighting matrices, $R, U,$ and Q reflect the policymaker's preferences; R and Q are assumed to be positive semidefinite and positive definite, respectively.

Acknowledging that their reference model may be misspecified, private agents and the policymaker surround their reference model with a class of models of the form

$$z_{1t+1} = A_{11} z_{1t} + A_{12} z_{2t} + B_1 u_t + C_1 (v_{t+1} + \varepsilon_{1t+1}), \tag{4}$$

$$E_t z_{2t+1} = A_{21} z_{1t} + A_{22} z_{2t} + B_2 u_t, \tag{5}$$

where v_{t+1} is a vector of specification errors, to arrive at a "distorted" model. The specification errors are intertemporally constrained to satisfy

$$E_0 \sum_{t=0}^{\infty} \beta^t v'_{t+1} v_{t+1} \leq \eta, \tag{6}$$

where $\eta \in [0, \overline{\eta}]$ represents the "budget" for missspecification.

Because private agents form expectations that are "rational" according to the distorted model, the non predetermined variables and their expected values are linked according to $z_{2t+1} = E_t z_{2t+1} + \varepsilon_{2t+1}$. The distorted model can be written as

$$z_{1t+1} = A_{11} z_{1t} + A_{12} z_{2t} + B_1 u_t + C_1 (v_{t+1} + \varepsilon_{1t+1}), \tag{7}$$

$$z_{2t+1} = A_{21} z_{1t} + A_{22} z_{2t} + B_2 u_t + \varepsilon_{2t+1}, \tag{8}$$

or, more compactly and in obvious notation, as

$$z_{t+1} = A z_t + B u_t + C v_{t+1} + \tilde{C} \varepsilon_{t+1}. \tag{9}$$

To guard against the worst case misspecification, the policymaker formulates policy subject to the distorted model with the view that the misspecification will be as damaging as possible. Private sector agents form expectations with the same view. The fear that the misspecification will be as damaging as possible is operationalized to the metaphor that v_{t+1} is chosen by an evil agent whose objectives are diametrically opposed to those of the policymaker. Hansen and Sargent (2001) prove that the constraint problem in which equation (3) is minimized with respect to $\{u_t\}_0^\infty$ and maximized with respect to $\{v_t\}_1^\infty$, subject to equations (9) and (6), can be recast in terms of an equivalent multiplier problem, whereby

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[z_t' R z_t + 2 z_t' U u_t + u_t' Q u_t - \theta v_{t+1}' v_{t+1} \right], \tag{10}$$

is minimized with respect to $\{u_t\}_0^{\infty}$ and maximized with respect to $\{v_t\}_1^{\infty}$, subject to equation (9). The parameter $\theta \in [\underline{\theta}, \infty)$ is a shadow price that is inversely related to the budget for misspecification η. Specifically, as η approaches 0, θ approaches infinity.

Robust policymaking with commitment using state – space methods

In the commitment solution, both the policymaker and the evil agent are assumed to commit to a policy strategy and not succumb to incentives to renege on that strategy. Employing the definitions

$$\tilde{u}_t \equiv \begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix}, \ \tilde{B} \equiv \begin{bmatrix} B & C_1 \end{bmatrix}, \tag{11}$$

$$\tilde{U} \equiv \begin{bmatrix} U & 0 \end{bmatrix}, \ \tilde{Q} \equiv \begin{bmatrix} Q & 0 \\ 0 & -\theta I \end{bmatrix} \tag{12}$$

the optimization problem can be written as

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[z_t' R z_t + 2 z_t' \tilde{U} \tilde{u}_t + \tilde{u}_t' \tilde{Q} \tilde{u}_t \right], \tag{13}$$

subject to

$$z_{t+1} = A z_t + \tilde{B} u_t + \tilde{C} \varepsilon_{t+1}, \tag{14}$$

which, because the first – order conditions for a maximum are the same as those for a minimum, has a form that can be solved using the methods developed by Backus and Drifill (1986). Those methods involve formulating the problem as linear – quadratic, the value function has the form $V(z_t) = z_t' V z_t + d$ and the dynamic program can be written as

$$z_t' V z_t + d = \min_{u_t} \max_{v_{t+1}} [z_t' R z_t + 2 z_t' \tilde{U} \tilde{u}_t' + \tilde{u}_t' \tilde{Q} \tilde{u}_t + \beta E_t (z_{t+1}' V z_{t+1} + d)]. \tag{15}$$

It is well known that the solution to this optimization problem takes the form

$$\begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix} = -F T^{-1} \begin{bmatrix} z_{1t} \\ p_{2t} \end{bmatrix}, \tag{16}$$

$$z_{2t} = [V_{22}^{-1}V_{21}V_{22}^{-1}]\begin{bmatrix} z_{1t} \\ p_{2t} \end{bmatrix}, \tag{17}$$

$$\begin{bmatrix} z_{1t+1} \\ p_{2t+1} \end{bmatrix} = T(A - \tilde{B}F)T^{-1}\begin{bmatrix} z_{1t} \\ p_{2t} \end{bmatrix} + C\varepsilon_{1t+1} \tag{18}$$

where p_{2t} is an $n_2 \times 1$ vector of shadow prices associated with the non predetermined variables, z_{2t}. The matrix T provides a mapping between the state variables, z_{1t} and p_{2t}, and z_t and is given by

$$T = \begin{bmatrix} I & 0 \\ V_{21} & V_{22} \end{bmatrix}, \tag{19}$$

where V_{21} and V_{22} are submatrices of V. Finally, V and F are obtained by solving for the fix – point of

$$V = R - 2\tilde{U}F + F'\tilde{Q}F + \beta(A - \tilde{B}F)'V(A - \tilde{B}F), \tag{20}$$

$$F = (\tilde{Q} + \beta\tilde{B}'V\tilde{B})^{-1}(\tilde{U}' + \beta\tilde{B}'VA). \tag{21}$$

When the worst case misspecification is realized, the economy behaves according to equations (16) – (18). While the worst case equilibrium is certainly interesting, it is also important to consider how the economy behaves when the reference model transpires to be specified correctly. Partitioning F into $[F_u' \quad F_v']'$ where F_u and F_v are conformable with u_t and v_{t+1}, respectively. Dennis (2005) shows that the approximating equilibrium has the form

$$z_{1t+1} = (A_{11} + A_{12}H_{21} + B_1F_{z1}^u)z_{1t} + (A_{12}H_{22} + B_1F_{p2}^u)p_{2t} + C_1\varepsilon_{1t+1}, \tag{22}$$

$$p_{2t+1} = M_{21}z_{1t} + M_{22}p_{2t}, \tag{23}$$

$$z_{2t} = H_{21}z_{1t} + H_{22}p_{2t}, \tag{24}$$

$$u_t = F_{z1}^u z_{1t} + F_{p2}^u p_{2t}, \tag{25}$$

where $H_{21} \equiv V_{22}^{-1}V_{21}$, $H_{22} \equiv V_{22}^{-1}$, $[F_{z1}^u \quad F_{p2}^u] \equiv -F_uT^{-1}$, and

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \equiv T(A - \tilde{B}F)T^{-1}. \tag{26}$$

Interestingly, the worst-case equilibrium and the approximating equilibrium share certain features. For instance, the worst-case equilibrium and the approximating equilibrium differ only with respect to the law of motion for the predetermined variables and, as a consequence, following innovations to the system the initial-period responses of the predetermined variables are the same for the approximating equilibrium as for the worst-case equilibrium. But since the decision rules for z_{2t} and u_t are also the same for the two equilibria, it follows that the initial-period responses by the nonpredetermined variables and by the policy variables are also the same. With respect to impulse response functions, differences between the approximating equilibrium and the worst-case equilibrium then only occur one period after innovations occur.

Furthermore, because the coefficient matrix on the innovations is C_1, which scales the standard deviations of the innovations, it follows that adding noise to the innovations or changing their correlation structure is not part of the evil agent's strategy. Instead, the optimally designed misspecification has the effect of changing the law of motion for the predetermined variables. More precisely, since the specification errors enter only the stochastic component of z_{1t}, the evil agent's strategy is to change the conditional means of the shock processes but not their conditional volatility. **As shown in Appendix A, these relationships between the worst-case and the approximating equilibria also hold under discretion.**

2.2 Robust policymaking with discretion using state – space methods

In the discretionary case, the optimization problem remains

$$\min_{\{u_t\}} \max_{\{v_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t [z_t' R z_t + 2 z_t' \widetilde{U} \widetilde{u}_t + \widetilde{u}_t' \widetilde{Q} \widetilde{u}_t], \tag{27}$$

subject to

$$z_{1t+1} = A_{11} z_{1t} + A_{12} z_{2t} + \widetilde{B}_1 \widetilde{u}_t + C_1 \varepsilon_{1t+1}, \tag{28}$$

$$E_t z_{2t+1} = A_{21} z_{1t} + A_{22} z_{2t} + \widetilde{B}_2 \widetilde{u}_t, \tag{29}$$

but now neither the policymaker nor the evil agent can commit. A convenient way to solve this dynamic optimisation problem is to apply the method presented by Backus and

Drifill (1986). Conjecturing that the solution for the non – predetermined variables in period $t+1$ has the form

$$z_{2t+1} = Hz_{1t+1}, \tag{30}$$

equations (28) – (30) imply that the non – predetermined variables, z_{2t}, depend on the predetermined variables, z_{1t}, and the control variables, $\tilde{u}_t$, according to

$$z_{2t} = Jz_{1t} + K\tilde{u}_t, \tag{31}$$

where

$$J \equiv (HA_{12} - A_{22})^{-1}(A_{21} - HA_{11}) \tag{32}$$

Using (31) to substitute the non – predetermined variables out of the objective function, the dynamic program for the optimisation problem with discretion is

$$z_{1t}'Pz_{1t} + k \equiv \min_{u_t} \max_{v_{t+1}} [z_{1t}'\overline{R}z_{1t} + 2z_{1t}'\overline{U}\tilde{u}_t + \beta E_t(z_{1t+1}'Pz_{1t+1} + k)], \tag{33}$$

where

$$\overline{R} \equiv R_{11} + R_{12}J + J'R_{21} + J'R_{22}J, \tag{34}$$

$$\overline{U} \equiv R_{12}K + J'R_{22}K + \tilde{U}_1 + J'\tilde{U}_2', \tag{35}$$

$$\overline{Q} \equiv K'R_{22}K + U_2'K + K'\tilde{U}_2 + \tilde{Q}, \tag{36}$$

and its solution given by

$$\begin{pmatrix} u_t \\ v_{t+1} \end{pmatrix} = -Fz_{1t}, \tag{37}$$

$$z_{2t} = (J - KF)z_{1t}, \tag{38}$$

$$z_{1t+1} = (A_{11} + A_{12}H - \tilde{B}_1F)z_{1t} + C_1\varepsilon_{1t+1} \tag{39}$$

where P and F are obtained by solving for the fix – point of

$$J \equiv (HA_{12} - A_{22})^{-1}(A_{21} - HA_{11}), \tag{40}$$

$$K \equiv (HA_{12} - A_{22})^{-1}(\tilde{B}_2 - H\tilde{B}_1), \tag{41}$$

$$\tilde{A}_{11} \equiv A_{11} + A_{12}J, \tag{42}$$

$$\tilde{A}_{12} = A_{12}K + \tilde{B}_1, \tag{43}$$

$$P = \overline{R} - 2\overline{U}F + F'\overline{Q}F + \beta(\tilde{A}_{11} - \tilde{A}_{12}F)'P(\tilde{A}_{11} - \tilde{A}_{12}F) \tag{44}$$

$$F = (\overline{Q} + \beta\tilde{A}_{12}'P\tilde{A}_{12})^{-1}(\overline{U}' + \beta\tilde{A}_{12}'P\tilde{A}_{11}), \tag{45}$$

$$H = J - KF \tag{46}$$

With the worst case equilibrium given by equations (37) – (39), partitioning F into $\begin{bmatrix} F_u' & F_v' \end{bmatrix}$ where F_u and F_v are conformable with u_t and v_{t+1}, respectively, the approximating equilibrium is derived from equations (37) – (39) by setting $F_v = 0$.

2.3 Robust policymaking with commitment using structural methods

While state-space solution methods have many advantages, being generally compact and containing only first-order dynamics, they are not always convenient. In particular, problems can arise from the fact that it is often difficult, sometimes prohibitively so, to manipulate a model into a state-space form, making state-space methods better suited to small models. But policymakers often employ medium- to large-scale models, and for this reason alone it is desirable to be able to solve robust control problems without relying on state-space methods. In this regard, Dennis (2006) has developed numerical methods that solve for optimal commitment policies and optimal discretionary policies in rational expectations models that allow the optimization constraints to be written in a structural form. These structural-form solution methods are easy to apply and offer considerable flexibility with regard to how the model is expressed.

One contribution of this section is to show that these structural-form methods can be readily applied to solve robust control problems. In fact, the advantages to using structural-form methods may extend somewhat further than convenience and flexibility. Leitemo and S¨oderstr¨om (2004, 2005) use a Lagrangian method—with the constraints in a structural form—to solve analytically for robustly optimal discretionary policies in closed- and open-economy models, respectively. They find that the evil agent's optimal strategy is to change the variances of the shocks, not their persistence, a strategy that differs from what the state-space methods outlined above would suggest.

In addition to illustrating how structural-form methods can be used to solve robust control problems numerically, we demonstrate that they need not generate the same worst case equilibrium as the state-space methods and explain why. We note that whereas with state-space methods the evil agent's strategy is to change the conditional means of the shocks, with structural-form methods the evil agent will generally choose to change both the conditional means and the variance/covariance structure of the shocks. As we show, these differences arise because the structural-form solution methods change slightly the nature of the game played between the agents in the model, accommodating a more general class of specification errors in the process. Finally, we outline how detection-error probabilities, essentially, the probability that an econometrician would make a model selection error, can be calculated given this more general class of specification errors.

The basic representation that Dennis (2006) works with is the second – order structural form. Therefore, let the reference model be represented as

$$A_0 y_t = A_1 y_{t-1} + A_2 E_t y_{t+1} + A_3 u_t + A_4 \varepsilon_t , \tag{47}$$

where y_t is an $n \times 1$ vector of endogenous variables, u_t is a $p \times 1$ vector of policy instruments, ε_t is an $s \times 1$, $0 \le s \le n$, vector of innovations, and A_0, A_1, A_2, A_3 and A_4 are matrices with dimensions conformable with y_t, u_t and ε_t that contain the structural parameters. The matrix A_0 is assumed to be nonsingular and the elements of A_4 are determined to ensure that shocks are distributed according to $\varepsilon_{1t} \approx iid[0, I_s]$. The dating on the variables is such that any variable that enters y_{t-1} is known by the beginning of period t; by construction the variables in y_{t-1} are predetermined. Binder and Pesaran (1995) show that this second – order structural form encompasses an enormous class of (log-) linear macroeconomic models.

With the reference model written in second – order structural form, private agents and the policymaker acknowledge their concern for misspecification by surrounding their reference model with a class of models of the form

$$A_0 y_t = A_1 y_{t-1} + A_2 E_t y_{t+1} + A_3 u_t + A_4 (v_t + \varepsilon_t), \tag{48}$$

where v_t is a vector containing specification errors and equation (48) represents the "distorted" model. Just as earlier, the specification errors are intertemporally constrained to satisfy:

$$E_0 \sum_{t=0}^{\infty} \beta^t v'_{t+1} v_{t+1} \le \omega , \tag{49}$$

where $w \in [0, \overline{w})$ represents the evil agent's total budget of misspecification.

The policy objective function is taken to be

$$E_0 \sum_{t=0}^{\infty} \beta^t [y_t' W y_t + u_t' Q u_t], \tag{50}$$

where $W(n \times n)$ and $Q(p \times p)$ are matrices containing policy weights and are symmetric positive semidefinite, and symmetric positive definite, respectively. Penalty terms on the interaction between y_t and u_t could be included, but are unnecessary because such terms can be accomodated through a suitable construction of y_t, reflecting the greater flexibility offered by the structural form.

Analogous to the state – space approach, the problem of minimizing equation (50) with respect to $\{u_t\}_0^{\infty}$ and maximizing with respect to $\{v_t\}_0^{\infty}$ subject to equations (48) and (49) can be replaced with an equivalent multiplier problem in which

$$E_0 \sum_{t=0}^{\infty} \beta^t [y_t' W y_t + u_t' Q u_t - \phi v_t' v_t], \tag{51}$$

is minimized with respect to $\{u_t\}_0^{\infty}$ and maximized with respect to $\{v_t\}_0^{\infty}$ subject to equation (48). The multiplier $\phi \in [\underline{\phi}, \infty)$ is inversely related to the budget for misspecification, ω. This method of formulating the robust control problem with the reference model and the distorted model in structural form parallels Hansen and Sargent (2006) closely. Nevertheless, we distinguish between ω and η and between ϕ and θ to acknowledge that ϕ and θ, while they are both shadow prices, need not share the same interpretation and that $\overline{\omega}$ and $\overline{\eta}$ need not take the same value.

To solve the robust control problem with commitment when the constraints are in second – order structural form, the optimization problem is formulated using the Lagrangian.

$$L = E_0 \sum_{t=0}^{\infty} \beta^t [y_t' W y_t + \tilde{u}_t' \tilde{Q} \tilde{u}_t + 2\lambda_t' (A_0 y_t - A_1 y_{t-1} - A_2 y_{t+1} - \tilde{A}_3 \tilde{u}_t - \rho_t)], \tag{52}$$

where

$$\tilde{Q} \equiv \begin{bmatrix} Q & 0 \\ 0 & -\phi I \end{bmatrix}, \ \tilde{A}_3 \equiv \begin{bmatrix} A_3 & A_4 \end{bmatrix}, \ \tilde{u}_t \equiv \begin{bmatrix} u_t \\ v_t \end{bmatrix}, \tag{53}$$

and $\rho_t \equiv A_4\varepsilon_t - A_2\varepsilon_t^y$, with $\varepsilon_t^y \equiv y_{t+1} - E_t y_{t+1}$. The first – order conditions with respect to $\tilde{u}_t$, λ_t, and y_t, respectively, can be written as

$$\frac{\partial L}{\partial \tilde{u}_t} = \tilde{Q}\tilde{u}_t - \tilde{A}_3'\lambda_t = 0, t \geq t_0 \tag{54}$$

$$\frac{\partial L}{\partial \lambda_t} = A_0 y_t - A_1 y_{t-1} - A_2 E_t y_{t+1} - \tilde{A}_3\tilde{u}_t - A_4\varepsilon_t = 0, t \geq t_0, \tag{55}$$

$$\frac{\partial L}{\partial y_t} = W y_t + A_0'\lambda_t - \beta^{-1}A_2'\lambda_{t-1} - \beta A_1' E_t\lambda_{t+1} = 0, t \geq t_0 \tag{56}$$

with the initial condition that $\lambda_{t-1} = 0$. Equations (54) – (56) describe a standard system of expectational equations, in which the expectations are formed rationally from the perspective of the distorted model and can be solved in a variety of ways. However this system is solved, the solution can be written as

$$\begin{bmatrix} \lambda_t \\ y_t \end{bmatrix} = \begin{bmatrix} H_{\lambda\lambda} & H_{\lambda y} \\ H_{y\lambda} & H_{yy} \end{bmatrix} \begin{bmatrix} \lambda_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} G_{\lambda\varepsilon} \\ G_{y\varepsilon} \end{bmatrix} \varepsilon_t, \tag{57}$$

$$\tilde{u}_t = \begin{bmatrix} F_\lambda & F_y \end{bmatrix} \begin{bmatrix} \lambda_{t-1} \\ y_{t-1} \end{bmatrix} + F_\varepsilon \varepsilon_t. \tag{58}$$

Equations (57) and (58) describe how the economy behaves in the worst – case equilibrium.

Given the worst case equilibrium, the approximating equilibrium, which is the equilibrium that pertains when the reference model is actually correctly specified, is

$$\lambda_t = H_{\lambda\lambda}\lambda_{t-1} + H_{\lambda y} y_{t-1} + G_{\lambda\varepsilon}\varepsilon_t, \tag{59}$$

$$u_t = F_\lambda^u \lambda_{t-1} + F_y^u y_{t-1} + F_\varepsilon^u \varepsilon_t, \tag{60}$$

$$\begin{aligned} y_t = {} & A_0^{-1}[A_1 + A_2(H_{y\lambda}H_{\lambda y} + H_{yy}H_{yy}) + A_3F_y^u]y_{t-1} + \\ & + A_0^{-1}[A_1 + A_2(H_{y\lambda}H_{\lambda\lambda} + H_{yy}H_{y\lambda}) + A_3F_\lambda^u]\lambda_{t-1} + \\ & + A_0^{-1}[A_4 + A_2(H_{y\lambda}G_{\lambda\varepsilon} + H_{yy}G_{y\varepsilon}) + A_3F_\varepsilon^u]\varepsilon_t \end{aligned} \tag{61}$$

Recall that for the state-space solution methods there were certain relationships between the worst-case equilibrium and the approximating equilibrium, relationships that held for both commitment and discretion. Specifically, the evil agent's strategy involved changing the persistence properties of the shocks, but not the volatility of the innovations, which meant that the initial period responses of the predetermined variables, the non-predetermined variables, and the policy controls to innovations would be the same for the worst-case equilibrium and the approximating equilibrium. Using the structural-form solution methods described above, however, these relationships do not necessarily hold.

To see this, note that the contemporaneous response of y_t to ε_t is $G_{y\varepsilon}$ in the worst – case equilibrium and $A_0^{-1}[A_4 + A_2(H_{y\lambda}G_{\lambda\varepsilon} + H_{yy}G_{y\varepsilon}) + A_3 F_\varepsilon^u]$ in the approximating equilibrium (see equation (61)). When these structural-form methods are employed, the evil agent's strategy may well involve a change to the variance-covariance matrix of the innovations as well as a change to the conditional means of the shock processes. It follows that the initial period responses by the endogenous variables, and hence also by the policy controls, to innovations may also differ between the worst-case and the approximating equilibria.

2.4 Robust policymaking with discretion using structural methods

In the discretionary environment, the optimization problem remains to

$$\min_{\{u_t\}_0^\infty} \max_{\{v_t\}_0^\infty} E_0 \sum_{t=0}^{\infty} \beta^t [y_t' W y_t + \tilde{u}_t' \tilde{Q} \tilde{u}_t] \tag{62}$$

$$A_0 y_t = A_1 y_{t-1} + A_2 E_t y_{t+1} + \tilde{A}_3 \tilde{u}_t + A_4 \varepsilon_t, \tag{63}$$

but, of course, neither the policymaker nor the evil agent can commit. The policymaker and the evil agent are Stackelberg leaders with respect to their future selves, but play a Cournot game between themselves. The problem described by equations (62) and (63) conforms to the class of problems studied and solved by Dennis (2006), where it is shown that the solution takes the form

$$y_t = H y_{t-1} + G \varepsilon_t, \tag{64}$$

$$\tilde{u}_t = F_1 y_{t-1} + F_2 \varepsilon_t. \tag{65}$$

The matrices H, G, F_1 and F_2 that govern the solution are arrived at through an iterative procedure. The first step involves conjecturing values for H and F_1 and using these to solve for the matrix D and the fix-point P according to

$$D \equiv A_0 - A_2 H \,, \tag{66}$$

$$P \equiv W + \beta F_1' \tilde{Q} F_1 + \beta H' P H \,. \tag{67}$$

Next, the values for D and P that solve equations (66) and (67) are used together with the conjectured values for H and F_1 to update F_1, F_2, H and G according to

$$F_1 = -(\tilde{Q} + \tilde{A}_3' D'^{-1} P D^{-1} \tilde{A}_3)^{-1} \tilde{A}_3' D'^{-1} P D^{-1} A_1 \,, \tag{68}$$

$$F_1 = -(\tilde{Q} + \tilde{A}_3' D'^{-1} P D^{-1} \tilde{A}_3)^{-1} \tilde{A}_3' D'^{-1} P D^{-1} A_4 \,, \tag{69}$$

$$H = D^{-1}(A_1 + \tilde{A}_3 F_1) \,, \tag{70}$$

$$G = D^{-1}(A_4 + \tilde{A}_3 F_2) \,. \tag{71}$$

From equations (68) – (71), updates of D and the fix – point P are generated, which in turn give rise to updated values for F_1, F_2, H and G. This iterative procedure continues until a fix – point in which F_1, F_2, H, G and P no longer change with successive iterations is obtained.

Equations (64) and (65) govern the economy's behavior in the worst-case equilibrium. From this worst-case equilibrium, the approximating equilibrium can be easily constructed; it is given by

$$y_t = A_0^{-1}[(A_1 + A_2 HH + A_3 F_1^u) y_{t-1} + (A_4 + A_2 HG + A_3 F_2^u)\varepsilon_t] \,, \tag{72}$$

$$u_t = F_1^u y_{t-1} + F_2^u \varepsilon_t \,, \tag{73}$$

where equation (72) exploits the fact that A_0 has full rank.

As one might expect, in the discretionary solution, just as in the commitment solution discussed above, the evil agent's strategy will generally involve changing both the persistence properties of the shocks and the variance-covariance matrix of the innovations. To see this, observe from equations (64) and (72) that the coefficient matrices on the innovations, G, and $A_4 + A_2 HG + A_3 F_2^u$, respectively, are not necessarily equal.

2.5 Detection – error probabilities

Anderson, Hansen, and Sargent (2003) describe the concept of a detection-error probability and introduce it as a tool for calibrating ϕ, the multiplier on the misspecification constraint, which would otherwise be a free parameter. A detection-error probability is the probability that an econometrician observing equilibrium outcomes would make an incorrect inference about whether the approximating equilibrium or the worst-case equilibrium generated the data. The intuitive connection between ϕ and the probability of making a detection error is that when ϕ is small, greater differences between the distorted model and the reference model (more severe misspecifications) can arise, which are more easily detected. Let A and B denote two models; with a prior that assigns equal weight to each model, Hansen, Sargent, and Wang (2002) show that detection – error probabilities are calculated according to

$$p(\phi) = \frac{prob(A \mid B) + prob(B \mid A)}{2}, \tag{74}$$

where $prob(A \mid B)$ ($prob(B \mid A)$) represents the probability that the econometrician errouneously chooses model A (model B) when in fact model B (model A) generated the data. Let model A denote the approximating model and model B denote the worstcase model, then any sequence of specification errors that satisfies equation (29) will be at least as difficult to distinguish from the approximating model as is a sequence that satisfies equation (29) with equality. As such, $p(\phi)$ represents a lower bound on the probability of making a detection error.

To calculate a detection-error probability we require a description of how the econometrician goes about choosing one model over another. Hansen, Sargent, and Wang (2002) assume that this model selection is based on the likelihood ratio principle. Let $\{z_t^B\}_1^T$ denote a finite sequence of economic outcomes generated according to the worst – case equilibrium, model B, and let L_{AB} and L_{BB} denote the likelihood associated with models A and B, respectively, then the econometrician chooses model A over model B if $\log(L_{BB}^n / L_{AB}^n) < 0$. Generating M independent sequences $\{z_t^B\}_1^T$, $prob(A \mid B)$ can be calculated according to

$$prob(A \mid B) \approx \frac{1}{M}\sum_{m=1}^{M} I\left[\log(L_{BB}^{n} / L_{AB}^{n}) < 0\right], \tag{75}$$

where $I[\log(L_{BB}^{n} / L_{AB}^{n}) < 0]$ is the indicator function that equals one when its argument is satisfied and equals zero otherwise; $prob(B \mid A)$ is calculated analogously using draws generated from the approximating model. The likelihood function that is generally used to calculate $prob(A \mid B)$ and $prob(B \mid A)$ assumes that the innovations are normally distributed.

While the theory of detection does not require that the evil agent not distort the volatility of the innovations, existing methods to calculate detection-error probabilities do (see Hansen, Sargent, and Wang, 2002, for example). Dennis, Leitemo and Soderstrom (2006) propose a more general method to calculate detection-error probabilities while accounting for the distortions to both the conditional means and the conditional volatilities of the shocks. Let

$$z_t = H_A z_{t-1} + G_A \varepsilon_t , \tag{76}$$

$$z_t = H_B z_{t-1} + G_B \varepsilon_t , \tag{77}$$

govern equilibrium outcomes under the approximating equilibrium and the worst – case equilibrium, respectively. With discretion, $z_t \equiv y_t$, while with commitment $z_t \equiv \begin{bmatrix} \lambda_t' & y_t' \end{bmatrix}'$. When $G_A \neq G_B$, to calculate $p(\phi)$ we must first allow for the stochastic singularity that generally characterizes equilibrium and second account appropriately for the Jacobian of transformation that enters the likelihood function. Using the QR decomposition we decompose G_A according to $G_A = Q_A R_A$ and G_B according to $G_B = Q_B R_B$. By construction, Q_A and Q_B are orthogonal matrices ($Q_A' Q_A = Q_B' Q_b = I_s$) and R_A and R_B are upper triangular. Let

$$\hat{\varepsilon}_t^{i|j} = R_i^{-1} Q_i' (z_t^j - H_i z_{t-1}^j), \quad \{i, j\} \in \{A, B\} \tag{78}$$

represent the inferred innovations in period t when model i is fitted to the data $\{z_t^j\}_1^T$ that are generated according to model j and let $\hat{\Sigma}^{i|j}$ be the associated estimates of the innovation variance – covariance matrices. Then

$$\log\left(\frac{L_{AA}}{L_{BA}}\right) = \log\left|R_A^{-1}\right| - \log\left|R_B^{-1}\right| + \frac{1}{2}tr(\hat{\Sigma}^{B|A} - \hat{\Sigma}^{A|A}), \tag{79}$$

$$\log\left(\frac{L_{BB}}{L_{AB}}\right) = \log\left|R_B^{-1}\right| - \log\left|R_A^{-1}\right| + \frac{1}{2}tr(\hat{\Sigma}^{A|B} - \hat{\Sigma}^{B|B}), \tag{80}$$

where tr is the trace operator.

When $G_A = G_B$ it follows that $R_A = R_B$ and the Jacobian of transformations associated with the various likelihoods cancel and play no role in the calculations, in which case equations (79) and (80) simplify to

$$\log\left(\frac{L_{AA}}{L_{BA}}\right) = \frac{1}{2}tr(\hat{\Sigma}^{B|A} - \hat{\Sigma}^{A|A}), \tag{81}$$

$$\log\left(\frac{L_{BB}}{L_{AB}}\right) = \frac{1}{2}tr(\hat{\Sigma}^{A|B} - \hat{\Sigma}^{B|B}), \tag{82}$$

which are equivalent to the expressions Hansen, Sargent, and Wang (2002) and Hansen and Sargent (2006, Chapter 8) employ. Given equations (79) and (80), equation (75) is used to estimate $prob(A|B)$ and (similarly) $prob(B|A)$, which are needed to construct the detection – error probability, as per equation (74). The multiplier, ϕ, is then determined by selecting a detection – error probability (or at least its lower bound) and inverting equation (74). Generally, this inversion is performed numerically by constructing the mapping between ϕ and the detection – error probability, for a given sample size.

2.6 Comparing the solution methods

We demonstrated that the solutions obtained for the worst-case equilibrium and the approximating equilibrium may depend on whether state-space methods or structural-form methods are used. Moreover, it should be clear that the differences between the two solution methods involve specification errors that are qualitatively different in important ways. For the structural-form solution methods, it is apparent that pessimistic agents are guarding against specification errors both to the conditional means of the shocks, which is the behavior Hansen and Sargent emphasize, and to the conditional variances/covariances of the shocks.

In an important sense, it is surprising that the solutions differ, as such differences do not arise when expectations are rational. But since the methods may produce different equilibrium behavior, two important questions immediately present themselves: why do the differences arise, and are the differences quantitatively important? With regard to the first question, when the solutions differ they do so because the state-space formulation restricts the various decisionmakers in ways that the structural-form formulation does not. In effect, the two methods are solving closely related, but not identical problems. To see this point, consider the following simple example. Let the reference model that the policymaker and private agents share be

$$y_t = \alpha E_t y_{t+1} + \gamma u_t + g_t, \tag{83}$$

$$g_t = \rho g_{t-1} + \sigma_\varepsilon \varepsilon_t, \tag{84}$$

where the parameters satisfy $\alpha \in (0,1), \gamma \in (-\infty,+\infty), \rho \in (-1,1)$, and $\sigma_g, \sigma_\varepsilon \in (0,\infty)$, and where ε_t is a mean – zero white – noise process with standard deviation equal to σ_ε. Notice that ε_t is an exogenous variable, u_t is a decision variable, y_t, $E_t y_{t+1}$, and g_t are non – predetermined variables, and g_{t-1} is a predetermined variable.

To write equations (83) and (84) in state-space form the standard method would be to advance the timing on equation (84) one period and to make $E_t y_{t+1}$ the subject of equation (83), giving

$$\begin{pmatrix} g_{t+1} \\ E_t y_{t+1} \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ -\frac{1}{\alpha} & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} g_t \\ y_t \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{\gamma}{\alpha} \end{pmatrix} (u_t) + \begin{pmatrix} \sigma_\varepsilon \\ 0 \end{pmatrix} (\varepsilon_{t+1}). \tag{85}$$

Adding the specification errors, the distorted model would then be

$$\begin{pmatrix} g_{t+1} \\ E_t y_{t+1} \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ -\frac{1}{\alpha} & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} g_t \\ y_t \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{\gamma}{\alpha} \end{pmatrix} (u_t) + \begin{pmatrix} \sigma_\varepsilon \\ 0 \end{pmatrix} (v_{t+1} + \varepsilon_{t+1}). \tag{86}$$

Notice that in equation (86) the shock g_t is a state variable, a variable that all agents take as given when forming decisions, even though it is not actually a predetermined variable.

In contrast, with the structural form method, once the model misspecifications are added to equation (64), becomes

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} g_t \\ y_t \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} E_t g_{t+1} \\ E_t y_{t+1} \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma \end{pmatrix}(u_t) + \begin{pmatrix} \sigma_\varepsilon \\ 0 \end{pmatrix}(v_t + \varepsilon_t) \quad (87)$$

In equation (87) the state variables that agents take as given when forming decisions are g_{t-1} and ε_t. Thus, the key difference between the two representations is that in the structural-form representation the state variables are g_{t-1}, which is predetermined, and ε_t, which is exogenous, while in the state-space representation the state variable is g_t,which is non-predetermined. Because the structural-form representation allows the evil agent to react separately to g_{t-1} and ε_t, if it so desires the evil agent can purposefully alter the realization of g_t, changing both the conditional mean of the shock and the variance of the innovation. Moreover, by allowing the specification errors to affect the contemporaneous realizations of the shocks, the nature of the strategic interaction between the policymaker and the evil agent is changed slightly by the structural form.

Before leaving this section, two final points are worth making. First, although the structural-form representation does not restrict the state vector, and permits a wider class of specification errors as a consequence, because all agents in the model—not just the evil agent—have their behavior restricted it is not the case that relaxing this restriction necessarily allows the evil agent to do more damage for a given budget. By relaxing the restriction, other agents in the economy can better guard against the specification errors. Second, state-space forms (and structural forms) are not unique. As a consequence, for any given model, a state-space representation that allows the evil agent to distort both the conditional mean and the conditional volatility of the shocks will generally be available.

2.7 Robust policy in an empirical model

To illustrate the robust control approach, we study the model estimated by Rudebusch (2002a), which is based on a standard New Keynesian model and contains two equations that, conditional upon the short – term interest rate, i_t, summarize the dynamics of inflation, π_t, and the dynamics of inflation, y_t:

$$\pi_t = \mu_\pi E_t \pi_{t+1} + (1-\mu_\pi)\pi_{t-1} + \alpha y_t + \varepsilon_{\pi,t}, \tag{88}$$

$$y_t = \mu_y E_t \pi_{t+1} + (1-\mu_y) y_{t-1} - \beta(i_t - E_t \pi_{t+1}) + \varepsilon_{y,t} \tag{89}$$

Equation (88) is a "New Keynesian Phillips curve" derived from the optimal pricesetting behavior of firms acting under monopolistic competition, but facing price rigidities, typically modeled following Calvo (1983). The presence of lagged inflation and the "supply shock" $\varepsilon_{\pi,t}$ can be motivated by indexing those prices that are not reoptimized in a given period and by a time-varying elasticity of substitution across goods, leading to time-varying markups. Equation (89) can be derived from the household consumption Euler equation, where habits in consumption imply that current decisions depend to some extent on past decisions. The "demand shock" $\varepsilon_{y,t}$ can be attributed to government spending shocks or to movements in the natural level of output.[8] An empirical version of this model, suitable for quarterly data and similar to that estimated by Rudebusch (2002a), is given by

$$\pi_t = \mu_\pi E_{t-1} \bar{\pi}_{t+3} + (1-\mu_\pi)\sum_{j=1}^{4} \alpha_{\pi j} \pi_{t-j} + \alpha_y y_{t-1} + \varepsilon_{\pi,t}, \tag{90}$$

$$y_t = \mu_y E_{t-1} y_{t+1} + (1-\mu_y)\sum_{j=1}^{2} \beta_{yj} y_{t-j} - \beta_r (i_{t-1} - E_{t-1}\bar{\pi}_{t+3}) + \varepsilon_{y,t} \tag{91}$$

where $\bar{\pi}_t = 1/4 \sum_{j=0}^{3} \pi_{t-j}$ is four – quarter inflation and i_t is the nominal federal funds rate (the policy instrument). We generalize the model slightly to include forward – looking behavior in the output gap equation, as in Rudebusch (2002b). The model's parameters estimates, shown in Table 1, are taken from Rudebusch (2002a) and are obtained using OLS (and survey expectations) on quarterly U.S. data from 1968:Q3 to 1996:Q4, except for the parameter μ_y, which is set to the average estimate in Fuhrer and Rudebusch (2004).

Table 1 – Parameter Values

Inflation		Output		Monetary Policy	
μ_π	0.29	μ_y	0.20	β	0.99
$\alpha_{\pi 1}$	0.07	β_{y1}	1.15	λ	0.50
$\alpha_{\pi 2}$	-0.14	β_{y2}	-0.27	υ	0.10
$\alpha_{\pi 3}$	0.40	β_r	0.09		
$\alpha_{\pi 4}$	0.07	σ_y	0.833		
α_y	0.13				
σ_π	1.012				

The model's key features are that inflation and the output gap are highly persistent, that monetary policy affects the economy only with a lag, and that expectations are formed using period $t-1$ information. Notice, also, that the weights on expected future inflation and output. While consistent with much of the empirical literature, are small relative to many theory – based specifications.

The central bank's objective function is assumed to be

$$\min_{\{i_t\}} E_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \lambda y_t^2 + v i_t^2), \tag{92}$$

where we $\beta = 0.99, \lambda = 0.5, v = 0.1$. Thus, the central bank sets monetary policy to avoid volatility in inflation around its target (normalized to zero) and in the output gap around zero (precluding any discretionary inflation bias). In addition, the central bank desires to limit volatility in the nominal interest rate around target (normalized to zero). The concern for misspecification, ϕ, is chosen so that the detection error probability is 0.1, given a sample of 200 observations. This implies that $\theta = 54.5$.

The model described by relations (90) – (92) can be written in state – space form as follows:

$$z_{t+1} = Az_t + Bu_t + C\varepsilon_{t+1}, \tag{93}$$

$$\min_{u_t} E_0 \sum_{t=0}^{\infty} \beta^t [z_t' R z_t + u_t' Q u_t], \tag{94}$$

where $z_{1t} = (\pi_t \quad \pi_{t-1} \quad \pi_{t-2} \quad \pi_{t-3} \quad y_t \quad y_{t-1})'$,

$z_{2t} = (E_t\pi_{t+1} \quad E_t\pi_{t+2} \quad E_t\pi_{t+3} \quad E_t y_{t+1})'$,

$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$,

$\varepsilon_t = (\varepsilon_{\pi t} \quad \varepsilon_{yt})'$,

$u_t = i_t$,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -6.56 & 1.37 & -3.92 & -0.69 & -1.79 & 0 & 12.79 & -1 & -1 & 0 \\ 0.74 & -0.15 & 0.44 & 0.077 & -4.4 & 1.08 & -1.44 & 0 & 0 & 0 \end{pmatrix},$$

$$B = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.45)'$$

$$C = \begin{pmatrix} 1.012 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.833 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$R=\begin{pmatrix}1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0.5&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{pmatrix},$$

$$Q=0.1$$

We first solved the linear quadratic optimization problem in the nonrobust case. The matrix which gives the optimal feedback is

$$K=\begin{pmatrix}20.1 & -10.36 & 3.54 & 0.913 & -6.947 & 2.399 & -41.088 & 40.245 & -16.047 & -4.296\end{pmatrix} \tag{95}$$

and the optimal control is:

$$u_t=i_t=Fz_t=-Kz_t\,. \tag{96}$$

Next, we solved the worst – case robust control problem. In this case,

$$u_t=\begin{pmatrix}i_t\\ v_{\pi,t+1}\\ v_{y,t+1}\end{pmatrix},$$

$$\overline{B}=\begin{pmatrix}0&1.012&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0.833\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0.45&0&0\end{pmatrix},$$

$$\overline{Q} = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 54.5 & 0 \\ 0 & 0 & 54.5 \end{pmatrix}.$$

Matrices *A*, *C* and *R* are the same as in the nonrobust case.

Solving the linear quadratic optimisation problem, we obtained the optimal feedback matrix

$$\overline{K} = \begin{pmatrix} 1.6733 & 0.992 & -1.65 & -0.302 & -9.741 & 2.3038 & -0.99 & 1.22 & -3.38 & -0.22 \\ -1.4045 & 0.783 & -0.61 & -0.125 & -0.315 & -0.0005 & 3.26 & -1.97 & 0.055 & 0.216 \\ -1.4969 & 0.472 & -0.83 & -0.152 & -0.422 & -0.0008 & 2.99 & -0.62 & -0.33 & 0.206 \end{pmatrix}$$

(97)

The optimal control is given by $\overline{u}_t = -\overline{K}z_t$, which means that the optimal policy rule and misspecification are given by:

Coefficient on

	π_t	π_{t-1}	π_{t-2}	π_{t-3}	y_t	y_{t-1}
Policy rule i_t	-1.6733	-0.992	1.65	0.302	9.741	0.99
Misspeficiation						
$v_{\pi,t+1}$	1.4045	-0.783	0.61	0.125	0.315	0.0005
$v_{y,t+1}$	1.4969	-0.472	0.83	0.152	0.422	0.0008

In figures 1, 2, we plot impulse responses to unit – sized innovations to inflation ($\varepsilon_{\pi,t}$) under commitment using the state – space method, for the nonrobust and robust cases, respectively.

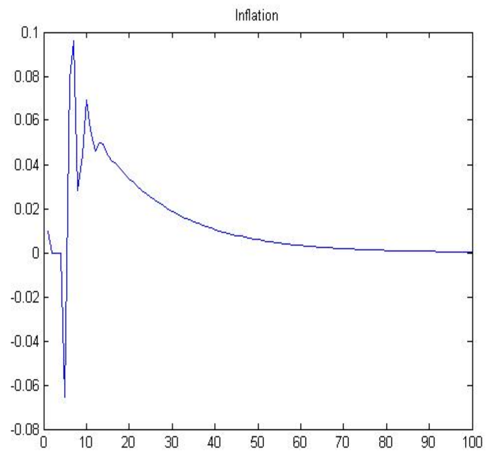

Figure 1

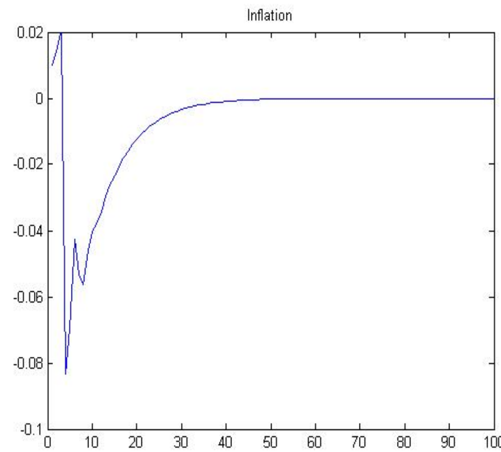

Figure 2

Figures 3 and 4 illustrate output responses to unit – sized innovations to inflation ($\varepsilon_{\pi,t}$) for the nonrobust and robust cases, respectively.

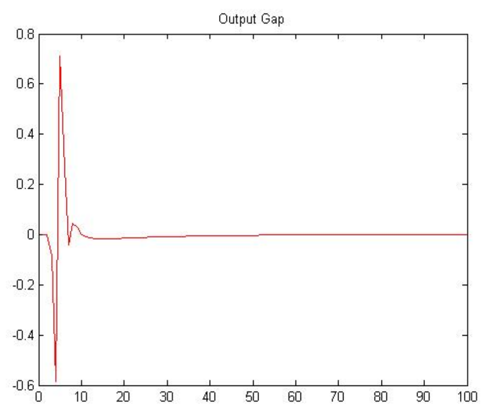

Figure 3

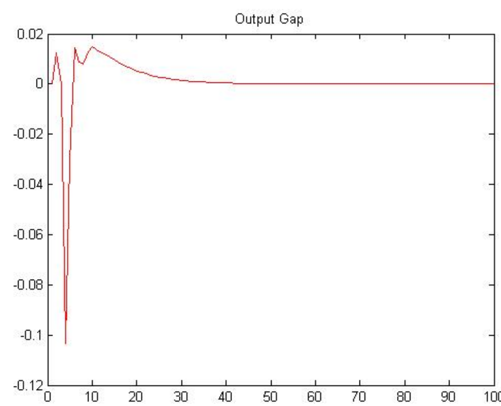

Figure 4

Figures 5 and 6 illustrate interest rate responses to unit – sized innovations to inflation ($\varepsilon_{\pi,t}$) for the nonrobust and robust cases, respectively.

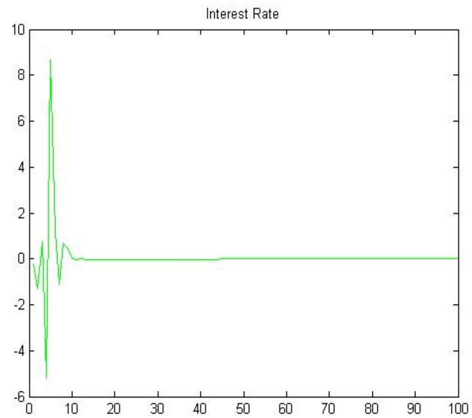

Figure 5

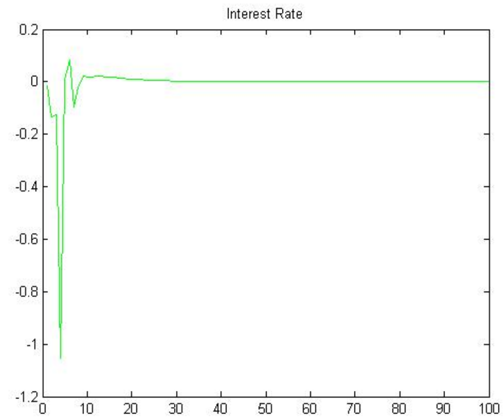

Figure 6

Under the nonrobust policy, a shock to inflation is followed by a prolonged period of high inflation, causing the central bank to tighten monetary policy and to raise the interest rate in order to open up a negative output gap, which will reduce inflation over time.

Using the state-space solution method in Figure 2, the misspecification has no effect in the initial period. In subsequent periods, however, the evil agent's actions, which make inflation more persistent in the worst-case equilibrium, produce a more aggressive policy response and a larger negative output gap and the effect on the output gap is considerably larger and more persistent. The more aggressive policy implies that the output gap is larger than under the nonrobust policy, and inflation therefore returns to target faster. Thus, the robust policy is more aggressive than the nonrobust policy, and the central bank fears mainly that inflation is more persistent than is reflected in the reference model. Giordani and Soderlind (2004) and Dennis, Leitemo and Soderstrom (2006) obtain qualitatively similar results.

Similar differences are obtained in response to output shocks (see Figures 6–12). Although the initial period distortion is small, the total effect is substantially larger and leads to quantitatively important differences between the two approaches.

Figures 7 and 8 illustrate inflation responses to unit – sized innovations to the output gap ($\varepsilon_{y,t}$) for the nonrobust and robust cases, respectively.

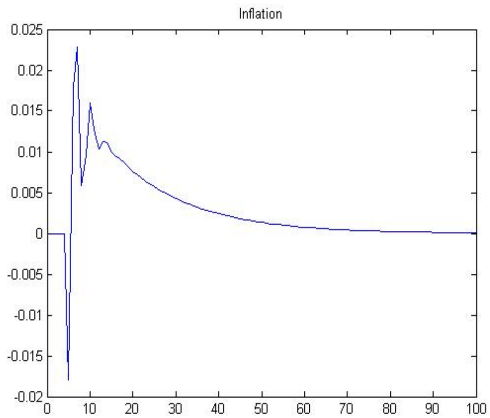

Figure 7

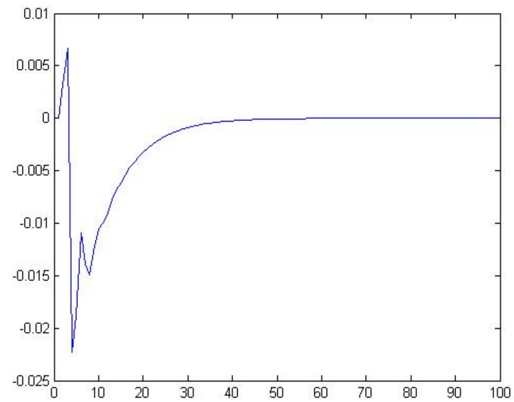

Figure 8

Figures 9 and 10 illustrate output gap responses to unit – sized innovations to the output gap ($\varepsilon_{y,t}$) for the nonrobust and robust cases, respectively.

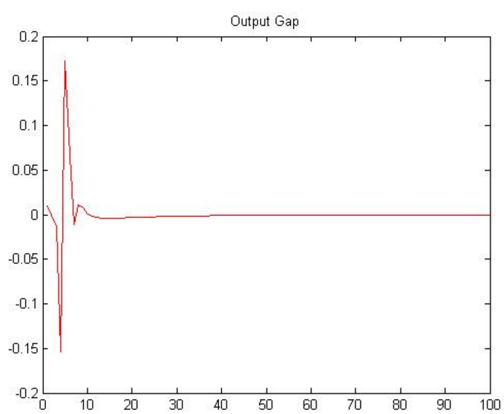

Figure 9

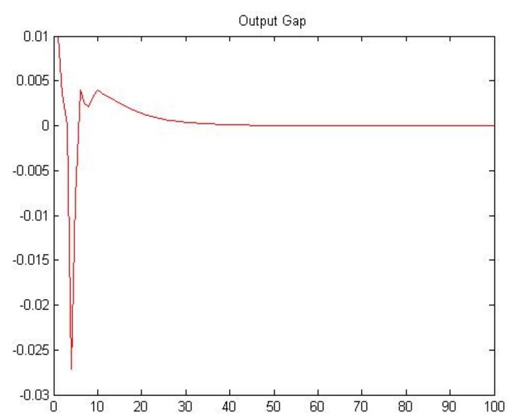

Figure 10

Figures 11 and 12 illustrate interest rate responses to unit – sized innovations to the output gap ($\varepsilon_{y,t}$) for the nonrobust and robust cases, respectively.

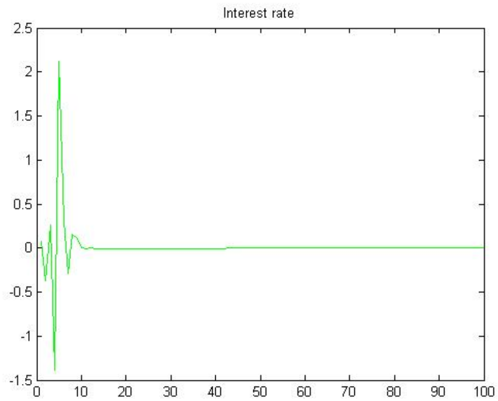

Figure 11

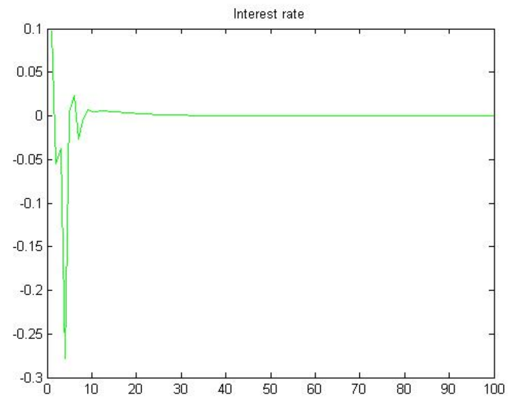

Figure 12

Conclusions

In formulating monetary policy, central banks must cope with substantial economic uncertainty.

Economic uncertainty can arise from different sources: the state of the economy, the nature of economic relationships, and the magnitude and persistence of ongoing shocks.

Robust control theory instructs decision makers to investigate the fragility of decision rules by conducting worst-case analyses.

In this paper we show how state space methods and structural-form solution methods can be applied to robust control problems, thereby making it easier to analyze complex models.

We illustrate the state space solution methods by applying them to an empirical New Keynesian business cycle model of the genre widely used to study monetary policy under rational expectations. A key finding from this exercise is that the strategically designed specification errors will tend to distort the Phillips curve in an effort to make inflation more persistent, and hence harder and more costly to stabilize. The optimal response to these distortions is for the central bank to become **more activist** in its response to shocks.

References

[1] Anderson, Evan, Lars Peter Hansen, and Thomas J. Sargent (2003), "A quartet of semigroups for model specification, robustness, prices of risk, and model detection", Journal of the European Economic Association 1 (1), 68–123.

[2] Backus, David K. and John Driffill (1986), "The consistency of optimal policy in stochastic rational expectations models", Discussion Paper No. 124, Centre for Economic Policy Research.

[3] Binder, Michael and M. Hashem Pesaran (1995), "Multivariate rational expectations models and macroeconometric modeling: A review and some new results" in Pesaran, M. Hashem and Michael R. Wickens (eds.), Handbook of Applied Econometrics. Volume 1 , Blackwell Publishers.

[4] Blanchard, Olivier J. and Charles M. Kahn (1980), "The solution of linear difference models under rational expectations", Econometrica 48 (5), 1305–1311.

[5] Calvo, Guillermo A. (1983), "Staggered prices in a utility-maximizing framework", Journal of Monetary Economics 12 (3), 383–398.

[6] Currie, David and Paul Levine (1985), "Optimal feedback rules in an open economy macromodel with rational expectations", European Economic Review 27 (2), 141–163.

[7] Currie, David and Paul Levine (1993), "Rules, Reputation and Macroeconomic Policy Coordination", Cambridge University Press.

[8] Dennis, Richard (2005), "Robust control with commitment: A modification to Hansen-Sargent", Working Paper No. 2005-20, Federal Reserve Bank of San Francisco.

[9] Dennis, R., Leitemo, K. and Soderstrom, U., “Methods for robust control”, Working Paper 2006 – 10, Federal Reserve Bank of San Francisco

[10] Engle, Robert F., David F. Hendry, and Jean-Francois Richard (1983), “Exogeneity”, Econometrica 51 (1), 277-304.

[11] Fuhrer, Jeffrey C. and Glenn D. Rudebusch (2004), “Estimating the Euler equation for output”, Journal of Monetary Economics, 51 (6), 1133–1153.

[12] Giordani, Paolo and Paul Soderlind (2004), “Solution of macromodels with Hansen-Sargent robust policies: Some extensions”, Journal of Economic Dynamics and Control 28 (12), 2367–2397.

[13] Hansen, Lars Peter and Thomas J. Sargent (2001), “Robust control and model uncertainty”, American Economic Review Papers and Proceedings 91 (2), 60–66.

[14] Hansen, Lars Peter and Thomas J. Sargent (2003), “Robust control of forward-looking models”, Journal of Monetary Economics 50 (3), 581–604.

[15] Hansen, Lars Peter and Thomas J. Sargent (2006), Robustness, University of Chicago and New York University, Princeton University Press.

[16] Hansen, Lars Peter, Thomas J. Sargent, and Thomas Tallarini (1999), “Robust permanent income and pricing,” The Review of Economic Studies 66 (4), 873–907.

[17] Hansen, Lars Peter, Thomas J. Sargent, and Neng Wang (2002), “Robust permanent income and pricing with filtering,” Macroeconomic Dynamics 6 (1), 40–84.

[18] Leitemo, Kai and Ulf Soderstr¨om (2004), “Robust monetary policy in the New-Keynesian framework,” Discussion Paper No. 4805, Centre for Economic Policy Research, Macroeconomic Dynamics.

[19] Leitemo, Kai and Ulf Soderstr¨om (2005), “Robust monetary policy in a small open economy,” Discussion Paper No. 5071, Centre for Economic Policy Research.

[20] Oudiz, Gilles and Jeffrey D. Sachs (1985), “International policy coordination in dynamic macroeconomic models”, in Buiter, Willem and Richard C. Marston (eds.), International Economic Policy Coordination, Cambridge University Press.

[21] Rudebusch, Glenn D. (2002a), “Assessing nominal income rules for monetary policy with model and data uncertainty”, Economic Journal, 112 (479), 1–31.

[22] Rudebusch, Glenn D. (2002b), “Term structure evidence on interest rate smoothing and monetary policy inertia,” Journal of Monetary Economics, 49 (6), 1161–1187.

[23] Woodford, Michael (2003), Interest and Prices: Foundations of a Theory of Monetary Policy, Princeton University Press.