ACADEMY OF ECONOMIC STUDIES
DOCTORAL SCHOOL OF FINANCE AND BANKING

CALIBRATION OF LIBOR MARKET MODEL

- comparison between the separated and the approximate approach –

- DISSERTATION PAPER -

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1. Abstract

This paper empirically analyzes and compares two methods of calibration for the Libor Market Models, developed by Brace, Gatarek and Musiela (1997) using data on EUR swaptions and historical EUR yield curves.

The first method of calibration proposed by Dariusz Gatarek is the separated approach, which gives good results but is computationally intensive. The second method of calibration – proposed by Ricardo Rebonato and Peter Jackel - uses an approximation for the instantaneous volatility and correlation functions of European swaptions in a forward rate based Brace-Gatarek-Musiela framework which enables us to calculate prices for swaptions without the need for Monte Carlo simulations. The method generates appropriate results in a fraction of a second.

To this end we show that using an approximation for the volatility and correlation function can lead to an accurate calibration by optimizing the parameters of the two volatility and correlation functions.

2. Introduction and Literature Review

The relatively brief history of the evolution of the pricing of interest – rate derivatives can be divided into four distinct periods as described by Rebonato [8]:

- **The early days** - The first period corresponds to the use of Black and Scholes (1973), Black (1976) and Merton (1973) approaches. They all have the same log-normal distributional assumption for the underlying variable (bond prices, forward rates, forward swap rates, bond yields). Criticism came because of the pull-to-par phenomenon (for a coupon or a discount bond the volatility is not constant – as argued in the three approaches - since the price has to converge to par at maturity). The solution was to consider a non-traded quantity as the underlying log-normal variable. However the Black-Scholes reasoning behind the self-financing dynamic strategy that reproduces the payoff of the option could not easily be adapted.

  The correct solution would have been to use the Black model instead of the Black-Scholes, with the forward price and the volatility of the forward as inputs. However, the Black formula does not include a correlation between prices of different assets.

  Despite being theoretically not justifiable, the approach was widely used for a long period as it allowed the trader to think in terms of volatility.

- **The first yield curve models** – Vasicek (1977) and Cox, Ingersoll and Ross (CIR) made the assumption that the dynamics of the whole yield curve would be driven by instantaneous short rate. The evolution was assumed to be described by a stochastic differential equation made up of deterministic mean-reverting component, and a stochastic part (proportional with the short rate itself or with the square root of the short rate).

  Despite the fact that the practical success of these models was limited, their influence was enormous, as all the models that were developed up until the HJM approach were part of the same, short-rate-based research programs.

The vast econometric research showed that the stochastic evolution of the yield curve is explained to a very large extent by its first principal component – the short rate being a reasonable proxy for this first component.

- **The Second-Generation Yield-Curve Models** – Black, Derman and Toy (1990), Hull and White (1990), extended Vasicek and extended CIR models. The most important feature of this class of models was the addition of a purely deterministic (time-dependent) term to the mean-reverting component in the drift of the short rate. Therefore, given an arbitrary yield market curve, the second-generation yield curve models could always augment the mean-reverting drift with a deterministic "correction term" capable of reproducing the market prices. The explanatory mandate of these models were transferred from accounting of the shape of the yield curve to assessing the reasonableness of the market term structure of volatilities. The class of market professionals that were still unsatisfied with the second-generation class of models were exotic traders (Bermudan swaptions, knock-out caps, inverse floaters, digitals) that requested a model able to price at least the required option hedges for each individual trade in line with the plain vanilla market.
- **The Modern Pricing Approach** – Heath-Jarrow-Morton (HJM). The HJM working paper began to be circulated as early as 1987, yet implementations started to appear around 1993-1994. This delay was encountered due to the relatively new language – set theory, measure theory, advanced stochastic calculus, linear partial differential equations – and to the non-Markovian structure of the log-normal forward rate process – therefore for the implementation the Monte Carlo method was required.

  The HJM model was originally cast in terms of instantaneous forward rates, wcih had no obvious equivalent in traded market instruments Furthermore, in the continuous –time limit and log-normal forward rates, their process explodes with positive probability. But, as soon as the process is discretized and the forward rates become of finite-tenor, the log-normal explosion disappears (Libor Market Model).

Due to various implementations of this model, there is no general agreement in the financial community on how to call this set of approaches: "BGM" (Brace, Gatarek and Musiela) model and "Jamishdian approach" are often used, but "pricing in a forward measure" or "the LIBOR Market Model" are terms also frequently encountered.

The calibration was seen to be an issue in the beginning, even if today it is agreed that one of the greatest advantage of the LIBOR-market model is that it can be made to reproduce the market prices of plain vanilla options. Any discrete time implementation of the HJM model is fully and uniquely specified by the instantaneous volatilities and instantaneous correlations among the discrete forward rates. Unfortunately, each of the possible choices for the instantaneous volatility functions would in general, give rise to different prices for exotic products. Furthermore, if this fitting was injudiciously carried out, it could produce implausible evolutions for such quantities as the term structure of volatilities or the swaption matrix.

It is essential to point out that the LIBOR market model as it is today, is much more than a set of equations for the no-arbitrage evolution of forward or swap rates: it includes a very rich body of calibration procedures and of approximate but very accurate numerical techniques for the evolution of forward rates that have turned the approach into the most popular pricing tool for complex interest-rate derivatives. Once the modern approach is properly implemented and calibrated, very complex computational tasks can be out with ease and in real trading time.

Until relatively recently, the calibration to market quantities of any interes-rate option model was one of the most arduous parts of its implementation. Users of early short-rate-based models (such as the Black-Derman and Toy (1990), the Hull and White or the Black-Karasinsky (1991)) are too well aware of the difficulties one encounters when attempting to calibrate the model parameters so as to reproduce the prices of caps or swaptions. Also the more recent Heath-Jarrow-Morton (HJM) approach is, in its more general form, hardly more user-friendly when it comes to calibration of the model to market data. The common features of all these models was the fact that, explicitly or implicitly, within these traditional frameworks the stochastic behavior was specified of *unobservable* financial quantities, such as, for instance, the instantaneous forward rates, the instantaneous short rate or its variance.

The calibration of a model to a set of market quantities therefore required transforming, via the black box provided by the model itself, the dynamics of these unobservable quantities into the dynamics of observable quantities.

The recently introduced Brace-Gatarek-Musiela (BGM) approach, germane to the HJM model, has radically changed this picture: now directly observable market quantities, such as discrete (LIBOR) forward rates or swap rates, are evolved. Given the availability from the market of the volatilities of caplets and European swaptions, calibration to *either* set of variables has become, at least for one-factor implementations, virtually immediate.

Implementation of the LMM basically consists of three parts, namely:

- **Calibration.** The calibration part adjusts the parameters of the LMM as to minimize the difference between LMM internal model values and actual prevailing market values. The user has to specify to which values should be calibrated. The calibration part requires market data.
- **Pricer.** The pricer part needs the time zero LIBOR forward rates, the parameters provided by the calibration part and it requires information from the derivative. The pricer part is either an analytic formula or a Monte Carlo (MC) simulation.
- **Derivative**-returns the derivative-payoff given a certain market scenario specified by the pricer part.

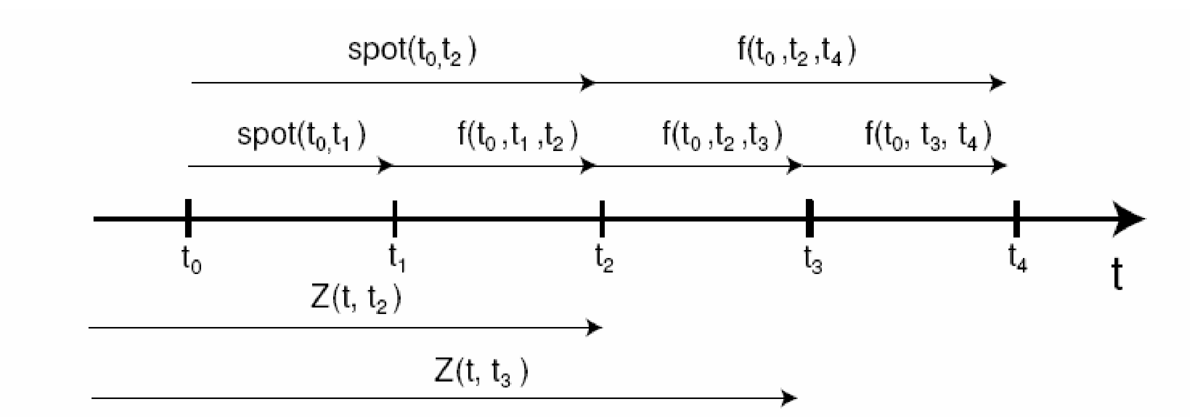
3.Description of the LIBOR Market Model

Overview

The LIBOR Market Model (LMM) is an interest rate model based on evolving LIBOR market forward rates. In contrast to models that evolve the instantaneous short rate (Hull-White, Black-Karasinski models) or instantaneous forward rates (Heath-Jarrow-Morton model), which are not directly observable in the market, the objects modeled using LMM are market-observable quantities (LIBOR forward rates). This makes LMM popular with market practitioners. Another feature that makes the LMM popular is that it is consistent with the market standard approach for pricing caps using Black's formula.

This chapter will begin exploring the LIBOR market model (LMM). It will start by describing the dynamics of the forward rates and determining the arbitrage free drift function.

The spot and forward rates for two forward rate structures with different tenor lengths and two of the corresponding zero coupon bonds.



The forward rates are defined as:

$$f(t, t_i, t_i + \tau_i)\tau_i = \frac{Z(t, t_i) - Z(t, t_i + \tau_i)}{Z(t, t_i + \tau_i)}$$

where $Z(t; t_i)$ is the price process of the zero-coupon discount bond that pays 1 at time t_i and τ_i is the tenor of the forward rate that resets at time t_i.

Forward rate dynamics in the LMM

The following notation will be used:

- $f_i(t)$ - Forward rate observed at time t for the period $t_i \rightarrow t_{i+1}$ with the compounding period $\tau_i = t_{i+1}-t_i$.
- $dW_k(t)$ - The k:th standard Brownian motion at time t.
- $\sigma_{ik}(t)$ - The instantaneous volatility function of the i:th forward rate for the k:th Brownian motion at time t
- μ_i - The drift parameter. Can depend on both time and on the forward rates themselves.

The forward rate dynamics is described by the m-dimensional diffusion equation:

$$\frac{df_i(t)}{f_i(t)} = \mu_i dt + \sum_{k=1}^{m} \sigma_{ik}(t) dW_k(t),$$

where the Brownian motions, $dW_k(t)$; $k = 1,\dots\dots\dots m$ are modelled as orthogonal i.e. the correlation between them is zero. The $\sigma_{ik}(t)$ s can be linked with the total volatility of the i:th forward rate. In order to do, when pricing swaptions and caplets, distinguish between the time-dependent instantaneous volatility for the forward rate resetting at time ti, $\sigma_i(t)$ and its implied “average” volatility given by the Black formula:

$$\sigma^2_{Black}(t_i)t_i = \int_0^{t_i} \sigma_i^2(s)ds.$$

In the above expression for the forward rate dynamics, $\sigma_{ik}(t)$ is denoted as the volatility contribution to the i_{th} forward rate given by the k_{th} Brownian motion. Using the standard formula for calculating the variance of the forward rate it is straightforward to show that the total instantaneous volatility of the forward rate $\sigma_i(t)$ and the $\sigma_{ik}(t)$ are related by:

$$\sigma_i^2(t) = \sum_{k=1}^{m} \sigma_{ik}^2(t).$$

Dividing and multiplying each loading, $\sigma_{ik}(t)$, with the instantaneous volatility of the i_{th} forward rate and using equation above gives:

$$\begin{aligned}\frac{df_i(t)}{f_i(t)} &= \mu_i dt + \sigma_i(t)\sum_{k=1}^{m}\frac{\sigma_{ik}(t)}{\sigma_i(t)}dW_k(t)\\ &= \mu_i dt + \sigma_i(t)\sum_{k=1}^{m}\frac{\sigma_{ik}(t)}{\sqrt{\sum_{k=1}^{m}\sigma_{ik}^2(t)}}dW_k(t)\\ &\equiv \mu_i dt + \sigma_i(t)\sum_{k=1}^{m} b_{ik}(t)dW_k(t)\end{aligned}$$

where $$b_{ik}(t) = \frac{\sigma_{ik}(t)}{\sqrt{\sum_{k=1}^{m}\sigma_{ik}^2(t)}}.$$

This formulation is very useful since it decomposes the orthogonal shocks of the forward rates into two distinct components. The first component, $\sigma_i(t)$ only depends on the total volatility of the ith forward rate. Also note that by definition:

$$\sum_{k=1}^{m} b_{ik}^2(t) = 1,$$

which implies that this component will not affect the caplet pricing at all and might instead be used to contain the models information about the correlation structure between the forward rates.

4. Calibrating the LIBOR MARKET MODEL

The calibration is the computation of the parameters of the LIBOR market model, σ_i, $i = 1 \ldots\ldots N$, so as to match as closely as possible model derived prices/values to market observed prices/values of actively traded securities.

Typically, a calibration procedure in a computer implemented LMM can take a few seconds up to fifteen minutes.

The wider meaning of calibration (Rebonato, 2002)

The meaning of the word ‘calibration’ has a much wider scope than that of just choosing the parameters of the model in such a way that today’s prices of the plain-vanilla instruments (caplets and swaps) are correctly recovered. This goal is important but limited, and only insures that the time-0 delta and vega hedging costs predicted by the model are the same as the ones provided in the market. The trader, however, will in general have to readjust the option hedges in his portfolio throughout the life of the deal. As long as the trader manages to recalibrate the model day after day to the future market prices, these re-hedging trades will always take place at the prices implied by the model at that point in time and at that state of the world.

It is important to point out that the common, and in practice unavoidable, procedure of recalibrating every day the model to the current market prices is essential. The practical success of a hedging strategy largely depends on the ability to choose, for a given model, a calibration, such that the parameters of the model have to be adjusted as little as possible throughout the life of the deal. This, in particular, will occur if the future realization of the term structure of volatilities and of the swaption matrix will be similar to the corresponding model-implied quantities.

4.1. The Separated approach with Optimization [4]

The Separated Approach provides a direct way of calibrating the model to the full set of swaptions.

We start our separated approach calibration by creating a matrix of Swaption Volatilities as below:

$$\Sigma^{SWPT} = \begin{pmatrix} \sigma_{1,2}^{swpt} & \sigma_{1,3}^{swpt} & \sigma_{1,4}^{swpt} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \sigma_{1,m+1}^{swpt} \\ \sigma_{2,3}^{swpt} & \sigma_{2,3}^{swpt} & \sigma_{2,4}^{swpt} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \sigma_{2,m+2}^{swpt} \\ \sigma_{3,4}^{swpt} & \sigma_{3,5}^{swpt} & \sigma_{3,6}^{swpt} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \sigma_{3,m+3}^{swpt} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sigma_{m,m+1}^{swpt} & \sigma_{m,m+2}^{swpt} & \sigma_{m,m+3}^{swpt} & \ldots\ldots\ldots\ldots\ldots & \sigma_{m,M}^{swpt} \end{pmatrix}$$

where we have:

$m=10$, $M=20$

$\sigma_{1,2}^{swpt} = \sigma^{swpt}(t, T_1,T_2)$ is the markwt swaption volatility for a swaption maturing at T_1 with underlying swap period $T_1 : T_2$

We can define the dependency of the components of Σ^{SWPT} on market swaption volatility symbols in the following way:

$$\sigma_{n,N}^{MKT} = \Sigma^{SWPT}(n,N-n)$$

Let us define the covariance matrix of Forward LIBOR rates in the following way:

$$\Phi^{I} = \begin{pmatrix} \varphi_{1,1}^{i} & \varphi_{1,2}^{i} & \varphi_{1,3}^{i} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \varphi_{1,m}^{i} \\ \varphi_{2,1}^{i} & \varphi_{2,2}^{i} & \varphi_{2,3}^{i} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \varphi_{2,m}^{i} \\ \varphi_{3,1}^{i} & \varphi_{3,2}^{i} & \varphi_{3,3}^{i} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \varphi_{3,m}^{i} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \varphi_{m,1}^{i} & \varphi_{m,2}^{i} & \varphi_{m,3}^{i} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \varphi_{m,M}^{i} \end{pmatrix}$$

where:

$$\varphi_{k,l}^{i} = \int_{0}^{T_i} \sigma^{inst}(t,T_{l-1},T_l) * \sigma^{inst}(t,T_{k-1},T_k)\, dt \text{ for } i<k \text{ and } i<l$$

and

$\sigma^{inst}(t,T_{l-1},T_l)$ is the stochastic instantaneous volatility of LIBOR rates $L_l(t,T_{l-1},T_l)$

We assume that:

$$\varphi_{k,l}^{i} = \varphi_{k,l} * \boldsymbol{\Lambda_i}$$

$$\boldsymbol{\Lambda_i = \delta_{0,k}\ ;\ k=1,\ldots..m}$$

$$\Phi = \begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} & \varphi_{1,3} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \varphi_{1,m} \\ \varphi_{2,1} & \varphi_{2,2} & \varphi_{2,3} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \varphi_{2,m} \\ \varphi_{3,1} & \varphi_{3,2} & \varphi_{3,3} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \varphi_{3,m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \varphi_{m,1} & \varphi_{m,2} & \varphi_{m,3} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \varphi_{m,M} \end{pmatrix}$$

Parameters on diagonal can be calculated using the formulae:

$$\varphi_{k,l} = [\boldsymbol{\delta_{0,k}} * \sigma^{swpt}(t, T_k, T_{k+1})^2] / \boldsymbol{\Lambda_k}$$

Computing $R_{i,j}^{k}$ (t) used for determining the non-diagonal elements of matrix Φ.

We define:

$$R_{i,j}^{k}(t) = \frac{B(0,T_{k-1}) - B(0,T_k)}{B(0,T_i) - B(0,Tj)}$$

where B are discount factors for the input LIBOR rates :

$$B = \begin{pmatrix} B(0,T_1) \\ B(0,T_2) \\ \ldots\ldots \\ B(0,T_M) \end{pmatrix}$$

Computing Φ:

We determine the non-diagonal parameters of our matrix Φ by using the following formulae:

$$\varphi_{k,N-1} = \frac{\boldsymbol{\delta_k} * \sigma_{k,N}^2 - \boldsymbol{\Lambda_k} \, \boldsymbol{(}\Sigma_{l=k+1}^{N} \Sigma_{i=k+1}^{N} * R_{k,N}^{i}(0) * \varphi_{i-1,l-1} * R_{k,N}^{l}(0) - 2 * R_{k,N}^{k+1}(0) * \varphi_{k,N-1} * R_{k,N}^{N}(0)}{2 * \boldsymbol{\Lambda_k} * R_{k,N}^{k+1}(0) * R_{k,N}^{N}(0)}$$

for k=1,…..m and N=k+2,……………

Computing modified matrix Φ:

Because it exists the risk for matrix Φ to have negative eigenvalues which might lead to some significant mispricing of European Swaptions we compute a modified matrix Φ^{PCA}, by removing the eigenvectors associated with negative eigenvalues.

Step 1:

We create a new matrix constructed by multiplying eigenvectors by correspondent square root of positive eigenvalues.

Step 2:

We multiply the matrix created in Step 1 by its transposition. In effect we obtain the modified matrix $\boldsymbol{\Phi^{PCA}}$**.**

Computing matrix of theoretical swaptions:

This matrix will contain theoretical swaption volatilities approximated via principal component modification of the initial covariance matrix.

$$\varphi_{k,l}^{PCA\ i} = \boldsymbol{\Lambda_i} * \varphi_{k,l}^{PCA}$$

and

$$\sigma_{kN}^2 \sim \Sigma_{l=k+1}^{N} \Sigma_{i=k+1}^{N} * R_{k,N}^{i}(0) * \varphi_{i-1,l-1}^{PCA} * R_{k,N}^{l}(0)$$

We define the root mean squared error between theoretical and market swaption volatilities as:

$$RMSE = \Sigma_{i,j=1}^{10} (\sigma_{i,j}^{THEO} - \sigma_{i,j}^{MKT})^2$$

Afterward, we add an optimization algorithm, setting as a target function the root mean squared error for the differences between theoretical and market volatilities. By this we determine the parameter vector Lambda that is obtained when we minimize RMSE function:

4.2. Approximate Solutions for Calibration [6] [8]

The main theory in this chapter is taken from the very rich text of Rebonato which provides a deep and thorough treatment of the whole calibration procedure.

The instantaneous volatility function

The calibration procedure intents to assure that the models instantaneous volatility function resembles the Black implied volatilities as good as possible.

In Rebonato several both parametric and non parametric functions for the instantaneous volatility function, $\sigma_i(t)$ are discussed.

σ_{inst} could be a deterministic function of:

- Calendar time: $\sigma_{inst}(t)$
- Specific feature of the forward rate itself, known exactly at time t for any τ>t, such as its maturity: $\sigma_{inst}(T)$
- Specific features of the foreard rate itself, whose future values are known at time t only in a statistical sense – the instantaneous volatility at time τ>t, could be made to depend for instance on the realization of the forward rate itself at time τ : $\sigma_{inst}(\tau ,T, f_{\tau})$
- The full history of the yield curve and/or of its stochastic drivers (jumps, diffusions) up to time t, as described, for instance, by the natural filtration F_t, generated by the evolution of this stochastic processes: $\sigma_{inst}(F_t)$
- The future realizations of the stochastic process, such as, for instance, Wiener processes, other than those driving the forward rates – in other terms, the instantaneous volatility could itself be a difussion, a jump process

Conditions for the volatility functions:

- Because we need to compute well-defined covariance elements, the volatility function must be square-integrable – the instantaneous volatility function should either belong to L^2 (the class of Lebesque-square integrable functions) or be square-integrable in the Riemann sense;

- The term-structure of volatilities (and/or of the swaption matrix should evolve in a time homogenous manner). If the instantaneous correlation is not a function of calendar time, a set of instantaneous-volatilities that produces a time-homogenous evolution of the term structure of volatilities will also give rise to similarly time-stationary swaption matrix;
- It should have a reasonably flexible functional form, so as to be able to reproduce either a humped or a monotonically decreasing instantaneous volatility;
- Its parameters should lend themselves to a reasonably transparent econometric interpretation, so as to allow a “sanity check” almost by inspection;
- It should afford easy analytical integration of its square, thereby greatly facilitating the evaluation of the necessary variance (and covariance) elements.

The functional form:

(A) $$\sigma_j(t) = k_j\left[(a + b(t_j - t))\,e^{-c(t_j-t)} + d\right]$$

fulfills this criteria to an acceptable degree

Putting k_j = 1 this form is clearly time-homogenous and displays, for suitable choices of the parameter set, a nicely humped term structure of volatility. However, the k_js allow a possibility for a perfect calibration in some cases and is therefore very useful. In order to preserve time-homogenousity it is, however, important to assure that the k_is are as close as possible to 1.

In order to preserve the short and long time behavior and the humped form of the termstructure of volatilities one may not choose the parameters *a*, *b*, *c* and *d* completely free.

For the interpretation of the function as a well behaved instantaneous volatility, the following conditions must be satisfied:

- $a + d > 0$
- $d > 0$
- $c > 0$.

Furthermore, when $\tau = t_j - t$ tends to zero instantaneous and average volatilities tend to coincide and therefore the quantity $a + d$ should at least approximately assume values given by the shortest maturities implied volatilities. On the other hand, when τ tends to large values d has to be connected with the very-long-maturity volatilities.

- $a + d$ ➔ Short maturities implied volatilities.
- d ➔ Very-long maturities implied volatilies.

Considering the first derivative of the time-homogeneous part of equation for the instantaneous volatility function with respect to τ: *(b-ca-cb τ/e^{-cr})* gives some final information:

- $(b - ca)/cb$ - The location of the extremum (the top of the hump).Should be > 0 and not too large
- $b > 0$ - Constraint for the extremum to be a maximum.

The instantaneous correlation function

This section will provide information about how to choose the model implied correlation between the forward rates. In general, this correlation function can be assigned a functional dependence on calendar times and on the maturities of the two forward rates: $\rho_{ij} = \rho\ (t, T_i, T_j)$.

In order for the covariance element $\int \rho_{ij}(t)\sigma_i(t)\sigma_j(t)dt$ to be well defined, once suitably square-integrable volatility functions have been chosen, it is enough to assign that the correlation function to be integrable over any interval $[T_k, T_{k+1}]$

However, the task of modeling instantaneous volatilities is considerably more complex for several reasons:

- the price of no plain-vanilla instrument depends purely on the correlation function and no other quantity;
- the correlation function always appears together with the opaque instanaeous colatility functions. Since these are in general time-dependent, this joint-occurrence makes the estimation of the correlation function from swaption prices even more difficult. Therefore it is difficult to disentangle the effects of the two contributions

If we assume that:

- the correlation function is time-homogenous, and
- it only depends on the relative distance in years between the two forward rates in question $(T_i - T_j)$

the form of the correlation function is:

$$\rho_{ij} = \rho\,(|T_i - T_j|)$$

It follows that : $\rho\,(|T_3-T_1|) = \rho\,(|T_3-T_2|) * \rho\,(|T_2-T_1|)$

In other terms, the logarithm of ρ must be a linear function. There must therefore exist some $\beta >= 0$ such that:

(B) $\rho_{ij} = \rho\,(|T_i-T_j|) = e^{-\beta|Ti-Tj|}$

If there is a need to price some heavy correlation-dependent derivatives the form above might not be good enough and more advanced market correlation providing functions have to be considered. Any feature more complex than that embodied by the equation above must come from either:

- movements of f_i uncorrelated with movements of f_j but correlated with movements of f_k
- a dependence of ρ other than on $|T_i-T_j|$

Given the parametrisation of the instantaneous volatility $\sigma_j(t)$ of the forward rate f_j and the correlation function given above, the indefinite integral of the covariance becomes:

(C)

$$\begin{aligned}
\int \rho_{ij}(t)\sigma_i(t)\sigma_j(t)dt \;=\;& e^{-\beta|t_i-t_j|}\frac{1}{4c^3}\cdot \\
&\cdot\Bigg(4ac^2d\left[e^{c(t-t_j)}+e^{c(t-t_i)}\right]+4c^3d^2t \\
&-4bcde^{c(t-t_i)}\left[c(t-t_i)-1\right]-4bcde^{c(t-t_j)}\left[c(t-t_j)-1\right] \\
&+e^{c(2t-t_i-t_j)}\Big(2a^2c^2+2abc\left[1+c(t_i+t_j-2t)\right] \\
&+b^2\left[1+2c^2(t-t_i)(t-t_j)+c(t_i+t_j-2t)\right]\Big)\Bigg)
\end{aligned}$$

Approximate solutions for calibration

In a forward-rate based BGM/J approach, once the time-dependent instantaneous volatilities and correlations of the forward rates have been specified, their stochastic evolution is completely determined.

Since swap rates are linear combinations (with stochastic weights) of forward rates, it follows that their dynamics are also fully determined once the volatilities and correlations of the forward rates have been specified. Some (very rare) complex derivatives depend exclusively on the volatility of either set of state variables (forward or swap rates).

In practical applications it is therefore extremely important to ascertain the implications for the dynamics of the swap rates, given a particular choice of dynamics for the forward rates and *vice versa*. Unfortunately, as shown later on, the *correct* evaluation of the swaption prices implied by a choice of forward rate volatilities and correlations is a conceptually straightforward, but computationally rather intensive exercise. The approximations presented below allow the estimation of a full swaption volatility matrix in a fraction of a second.

The no-arbitrage evolution of the forward rates is specified by the choice of a particular functional form for the forward-rate instantaneous volatilities and for the forward-rate/forward-rate correlation function (as described in the previous section).

The task at this point is to obtain the corresponding swap-rate instantaneous volatilities. Let $\sigma^{N*M}(t)$ denote the relative instantaneous volatility at time t of a swap rate SR^{N*M} expiring N years from today and maturing M years thereafter. This swap rate can be viewed as depending on the forward rates of that part of the yield curve in an approximately linear way:

(1)
$$SR^{N\times M}(t) = \sum_{i=1}^{n} w_i f_i(t)$$

with the weights wi given by:

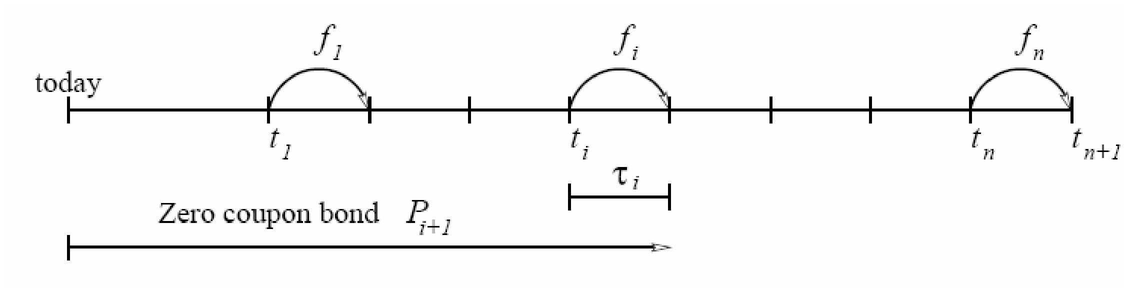
(2)
$$w_i = \frac{P_{i+1}\tau_i}{\sum_{k=1}^{n} P_{j+1}\tau_j}$$

- P_{i+1} -zero coupon bond maturing at the payment time of the i-th forward rate fi
- τ_i - the associated accrual factor

such that:

(3) $$P_{i+1} = \left[\prod_{k=1}^{i}(1 + f_k\tau_k)\right]^{-1} \cdot P_1$$

- n is the number of forward rates in the swap as illustrated schematically below:



- t_1 := N and t_{n+1} := N +M.

A straightforward application of Itˆo's lemma gives:

(4) $$\left[\sigma^{N\times M}\right]^2 = \frac{\sum_{j,k=1}^{n} w_j w_k f_j f_k \rho_{jk}\sigma_j\sigma_k}{\left[\sum_{i=1}^{n} w_i f_i\right]^2}$$

- $\sigma_j(t)$ is the time-t instantaneous volatility of log-normal forward rate f_j
- $\rho_{jk}(t)$ is the instantaneous correlation between forward rate f_j and f_k.

Expression (4) shows that the instantaneous volatility at time t > 0 of a swap rate is a stochastic quantity, depending as it does on the coefficients w, and on the future realization of the forward rates underlying the swap rate f.

Insofar as the weights w are concerned, which are functions of discount factors, one might be tempted to claim that their volatility should be very low compared to that of the swap or forward rates, and, as such, negligible. The same argument, however, certainly cannot be made about the forward rates themselves that enter equation (4). One therefore reaches the conclusion that, starting from a purely deterministic function of time for the instantaneous volatilities of the forward rates, one arrives at a rather complex, and *stochastic*, expression for the instantaneous volatility of the corresponding swap rate.

Therefore, in order to obtain the price of a European swaption corresponding to a given choice of forward-rate instantaneous volatilities, one is faced with a computationally rather cumbersome task: to begin with, in order to obtain the total Black volatility of a given European swaption to expiry, in fact, one first has to integrate its swap-rate instantaneous volatility:

(5)
$$\left[\sigma_{\text{Black}}^{N\times M}\right]^2 \cdot t_1 \quad = \quad \int_{u=0}^{t_1} \left[\sigma^{N\times M}(u)\right]^2 du$$

with t_1 being the time horizon of expiry of the option in N years from today as defined before.

By equation (4) one can conclude that, starting from a purely *deterministic* volatility for the (logarithm of) the forward rates, the instantaneous volatility of the corresponding swap rate is a stochastic quantity, and that the quantity is a path-dependent integral that cannot be equated to the (path-independent) real number

Calculating the value of several European swaptions, or, perhaps, of the whole swaption matrix, therefore becomes a very burdensome task, the more so if the coefficients of the forward-rate instantaneous volatilities are not given *a priori* but are to be optimised via a numerical search procedure so as to produce, say, the best possible fit to the swaption market.

Some very simple but useful approximations are however possible. In order to gain some insight into the structure of equation (4), one can begin by regarding it as a weighted average of the products $\rho_{jk}(t)\sigma_j(t)\ \sigma_k(t)$ with doubly-indexed coefficients $\zeta_{jk}(t)$ given by:

(6)
$$\zeta_{jk}(t) \quad = \quad \frac{w_j(t)f_j(t)\,w_k(t)f_k(t)}{\left[\sum_{i=1}^{n} w_i(t)f_i(t)\right]^2}$$

to convert equation (4) to:

(7)
$$\left[\sigma^{N\times M}(t)\right]^2 \quad = \quad \sum_{j,k=1}^{n} \zeta_{jk}(t)\rho_{jk}(t)\sigma_j(t)\sigma_k(t)$$

Approximating the $\zeta_{jk}(t)$:

For a given point in time, and for a given realization of the forward rates, these coefficients are, in general, far from constant or deterministic. Their stochastic evolution is fully determined by the evolution of the forward rates.

We distinguish two important cases:

Case 1: refers to (proportionally) parallel moves in the yield curve.

Each individual weight is only mildly dependent on the stochastic realization of the forward rate at time t. Intuitively this can be understood by observing that a given forward rate occurs both in the numerator and in the denominator of equation (6). So the effects on the coefficients of a (reasonably small) identical proportional change in the forward rates to a large extent cancel out.

Case 2: occurs when the yield curve experiences more complex changes.

For more complex changes in the shape of the yield curve, the individual coefficients remain less and less constant with increasing order of the principal component. In the less benign case of tilts and bends in the forward curve, the difference between the coefficients calculated with the initial values of the forward rates and after the yield curve move will in general be significant. However, in these cases one observes that the *average* of each individual weight corresponding to a positive and negative move of the same magnitude is still remarkably constant.

Note that the second statement has wider applicability (it does not require that the forward curve should only move in parallel), but yields weaker results, only referring as it does to the *average* of the instantaneous volatility. Note also that the average of the weights over symmetric shocks becomes less and less equal to the original weights as the complexity of the deformation increases; on the other hand we know that relatively few principal components can describe the yield curve dynamics to a high degree of accuracy.

Therefore, the negative impact of a progressively poorer approximation becomes correspondingly smaller and smaller.

Conclusions

1. To the extent that the movements in the forward curve are dominated by a first (parallel) principal component, the coefficients ζ are only very mildly dependent on the path realizations.
2. Even if higher principal components are allowed to shock the forward curve, the *expectation* of the future swap rate instantaneous volatility is very close to the value obtainable by using today's values for the coefficients ζ and the forward rates f.

1. Even if higher principal components are allowed to shock the forward curve, **the expectation of *the average Black volatility* is very close to the value obtainable by integrating the swap rate instantaneous volatilities calculated using today's values for the coefficients ζ and the forward rates f.**

It is well known, on the other hand, that the price of an at-the-money plain-vanilla option, such as a European swaption, is to a very good approximation a linear function of its implied Black volatility. This conclusion, by itself, would not be sufficient to authorize the trader to quote as *the* price for the European swaption the (approximate) average over the price distribution. The dispersion of the swaption prices around their average is however very small. If one therefore assumes that swaptions and forward rates can have simultaneously deterministic volatilities, and makes use of the results in about the likely impact of the joint log-normal assumption, it is possible to engage in a trading strategy that will produce, by expiry, imperfect but very good replication.

Therefore the expression:

(8)
$$\left[\sigma^{N\times M}(t)\right]^2 \simeq \sum_{j,k=1}^{n} \zeta_{jk}(0)\rho_{jk}(t)\sigma_j(t)\sigma_k(t)$$

should yield a useful approximation to the instantaneous volatility of the swap rate, and, ultimately, to the European swaption price.

It is essential to note that the above equation differs subtly but fundamentally from equation (7) in that the coefficients ζ are no longer stochastic quantities, but are evaluated using today's known values for the forward rates and discount factors.

By virtue of the previous results on the *average* of the ζ coefficients, a robust approximation for the equivalent implied Black volatility of a European swaption can be derived since the risk-neutral price of an option is given by the *expectation*, i.e. the *average* over the risk-neutral measure. The expression for the average Black volatility then becomes

(9)
$$\sigma_{\text{Black}}^{N\times M} = \sqrt{\sum_{j,k=1}^{n} \zeta_{jk}(0) \int_{u=0}^{1} \sigma_j(ut_1)\sigma_k(ut_1)\rho_{jk}(ut_1)\, du}$$

Equation (9) should be very useful in the context of calibration of FRA-based BGM/J models to market given European swaption volatilities. It enables us to calculate prices for the whole swaption matrix *without having to carry out a single Monte Carlo simulation* and thus to solve the highly cumbersome problem of calibration with great ease.

As shown in the result section, the quality of this approximation is very good. In those situations (noticeably non-flat yield curves) where it begins to prove unsatisfactory, it can be easily improved upon by a natural extension, which is presented in the next section.

Refining the approximation

The application of Itˆo's lemma to equation (1) actually gives equation (4) only if one assumes that the weights w are independent of the forward rates f. More correctly, Itˆo's lemma gives:

(10)
$$\begin{aligned}\frac{dSR}{SR} &= \sum_{i=1}^{n} \frac{\partial SR}{\partial f_i} \cdot \frac{df_i}{SR} \\ &= \sum_{i=1}^{n} \frac{\partial SR}{\partial f_i} \cdot \frac{f_i}{SR} \cdot \sigma_i d\widetilde{W}_i\end{aligned}$$

where in the Wiener processes Wi are correlated:

(11)
$$\left\langle d\widetilde{W}_i \cdot d\widetilde{W}_j \right\rangle = \rho_{ij} dt \ .$$

Given the definition

(12)
$$A_i = \sum^{n} P_{j+1} f_j \tau_j$$

of co-terminal floating-leg values and

(13)
$$B_i = \sum_{j=i}^{n} P_{j+1} \tau_j$$

for the co-terminal fixed-leg annuities, we obtain after some algebraic manipulations

(14)
$$\frac{\partial SR}{\partial f_i} = \left\{ \frac{P_{i+1}\tau_i}{B_1} - \frac{\tau_i}{1+f_i\tau_i} \cdot \frac{A_i}{B_1} + \frac{\tau_i}{1+f_i\tau_i} \cdot \frac{A_1 B_i}{B_1^2} \right\} \ .$$

This enables us to calculate the following improved formula for the coefficients ζ :

(15)
$$\zeta_{ij} \ = \ \left[\underbrace{\frac{P_{i+1} f_i \tau_i}{A_1}}_{\text{as in equation (6)}} + \underbrace{\frac{(A_1 B_i - A_i B_1) f_i \tau_i}{A_1 B_1 (1 + f_i \tau_i)}}_{\text{shape correction}} \right] \cdot \left[\underbrace{\frac{P_{j+1} f_j \tau_j}{A_1}}_{\text{as in equation (6)}} + \underbrace{\frac{(A_1 B_j - A_j B_1) f_j \tau_j}{A_1 B_1 (1 + f_j \tau_j)}}_{\text{shape correction}} \right]$$

We call the second term inside the square brackets of equation (15) the *shape correction*. Rewriting this corrective term as:

(16)
$$\frac{(A_1 B_i - A_i B_1) f_i \tau_i}{A_1 B_1 (1 + f_i \tau_i)} = \frac{f_i \tau_i}{A_1 B_1 (1 + f_i \tau_i)} \cdot \sum_{l=1}^{i-1} \sum_{m=i}^{n} P_{l+1} P_{m+1} \tau_l \tau_m (f_l - f_m)$$

highlights that it is a weighted average over inhomogeneities of the yield curve. In fact, for a flat yield curve, all of the terms (f_l - f_m) are identically zero and the righ-hand-side of equation (15) is identical to that of equation (6).

5. Data

The data for this study consists of daily yield curves for a trading period 20-Jun-2007 – 20-Jun-2008 across 40 maturities between 1 month and 50 years. Every series contains 254 observations. The data was obtained from Reuters 3000Xtra. This data serves for a basis for the volatility function estimation. The time series are obtained from different portions of the yield curve and from different quoted instruments:

- LIBOR cash deposit rates at the very short end: 1M, 2M, 3M (1month, 2 months, 3 months)
- Future contracts for intermediate maturities: H, M, U, Z (March, June, September, December)
- Equilibrium (par) swap rates for expiries between two years and the end of the LIBOR curve: 2S →50S (2 years – 50 years)

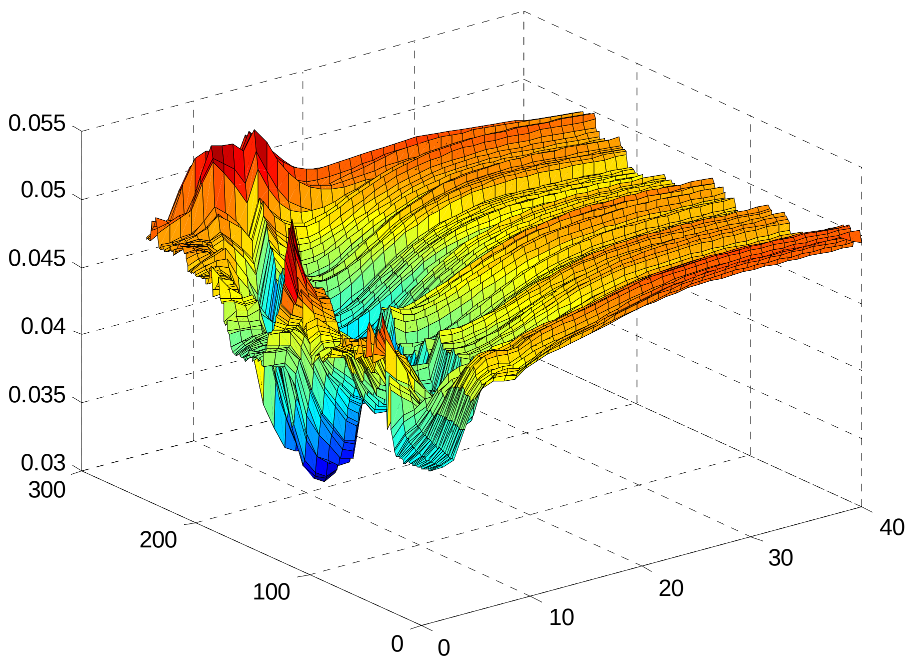


Figure 1: Historical yield curve dynamics 20-Jun-2007 → 20-Jun-2008

The correlation matrix obtained for the forward rate curves derived from the initial yield curve is presented in Figure 2 below:

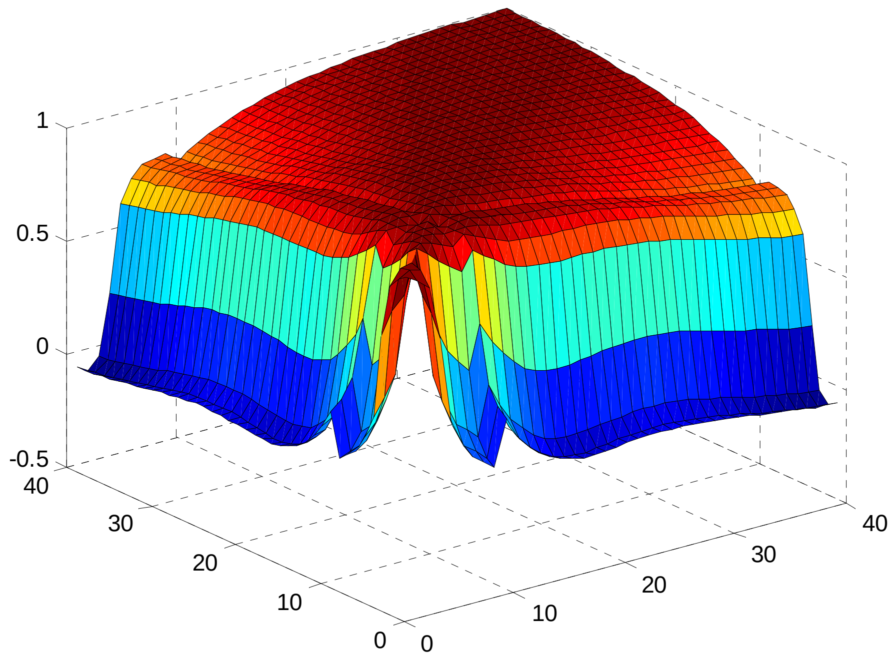


Figure 2:

Correlation of forward rates

The market volatility matrix used for comparison in the calibration methods consists of Black implied volatilities of ATM European swaptions. The data was obtained from Reuters 3000Xtra, being provided by ICAP Brokers.

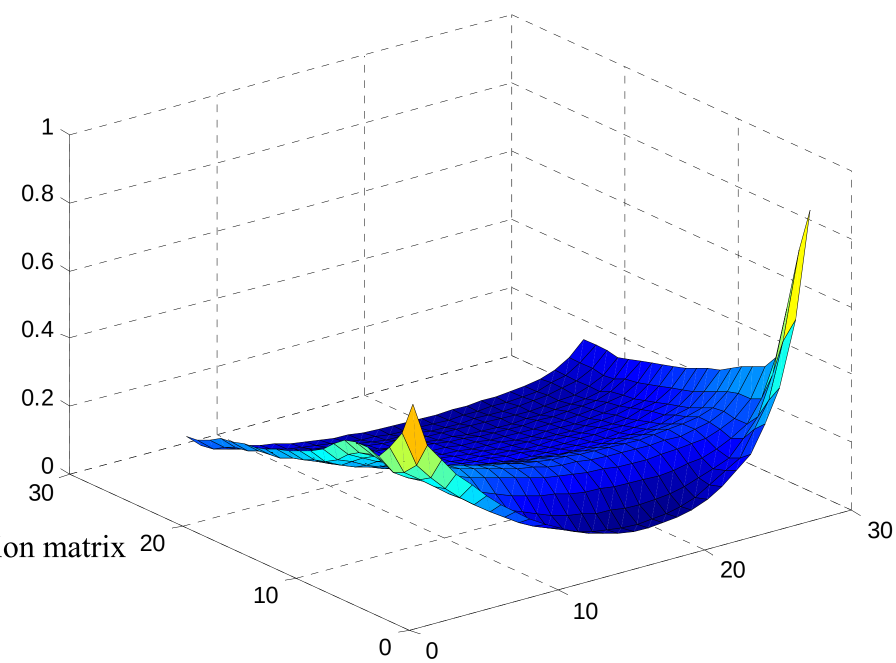


Figure 3:

Market swaption matrix

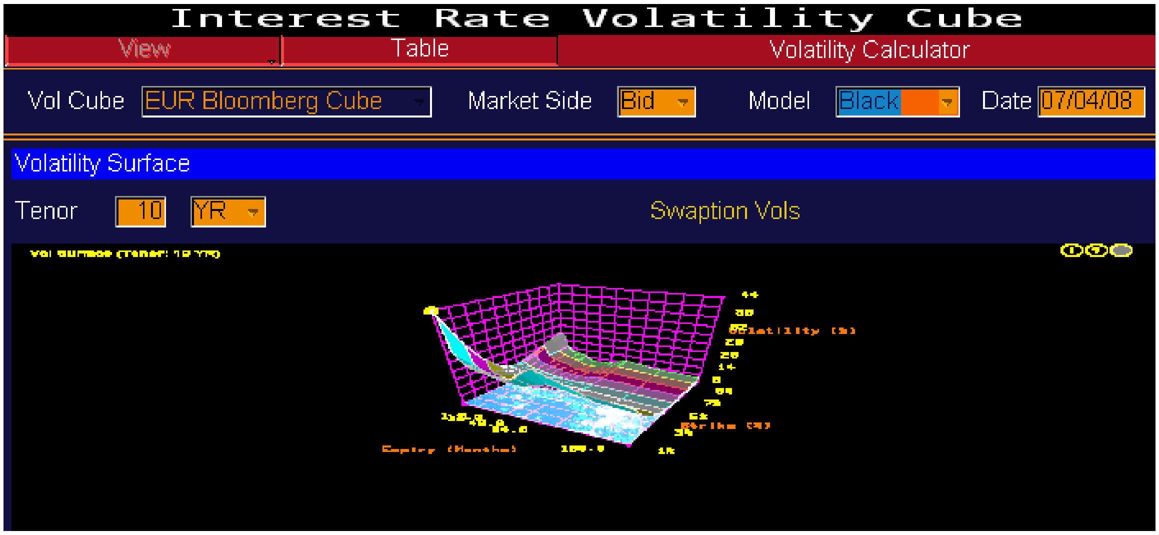


Figure 4: 10Y swaption volatilities as of July 4th, 2008 – Source: Bloomberg

As the first calibration procedure is computationally intensive, I reduced the data to a 10 maturities yield curve (from 1 year to 10 years) and a (10,10) matrix for swaption quoted Black volatilities. In order to ensure comparability between the two calibration methods I used the same set of data for the second calibration tool.

The calibration routines are created in Matlab.

5.1. Results - The Separated approach with Optimization

The steps for the calibration algorithm:

1. Input the initial data for the calibration:
    - Matrix of market swaption volatilities
    - Vector of dates and discount factors obtained for the current yield curve
2. Define the variance-covariance matrix of the forward of LIBOR Rates.
3. Transform the obtained matrices to ensure positivity of the matrices. For this I utilize the Principal Component Analysis (described in Appendix). For that case we implement a sub-algorithm for reducing the VCV matrix by removing eigenvectors associated with negative eigenvalues.

4. Add an optimization algorithm. The target function is the root mean squared error for the difference between the theoretical and market swaption volatilities.
5. Minimize that function under several restrictions for VCV and obtain the specification of parameters Λ_i used in the calibration

Code in Matlab used for the calibration as well as calibration results can be found in Appendix.

If we analyse the eigenvalues and eigenvectors, we can see that we have obtained only two negative eigenvalues and of very small absolute value. However, their explanatory power is very low. In the table below is the value of the eigenvalue with the explanatory power in the model:

L (vector of eigenvalues)	Explanatory power	Cumulative explanatory power
-0.077352	-0.043544311	-0.043544311
-0.037983	-0.02138204	-0.06492635
0.0076164	0.004287554	-0.060638796
0.060573	0.034098789	-0.026540007
0.084643	0.047648685	0.021108678
0.11274	0.063465529	0.084574206
0.24182	0.136129449	0.220703656
0.28975	0.163111025	0.38381468
0.46193	0.260037534	0.643852215
0.63266	0.356147785	1

Table 1. Explanatory Power of Eigenvalues

The first four biggest eigenvalues have much bigger values than other eigenvalues. The eigenvectors associated with the first four biggest eigenvectors are presented below:

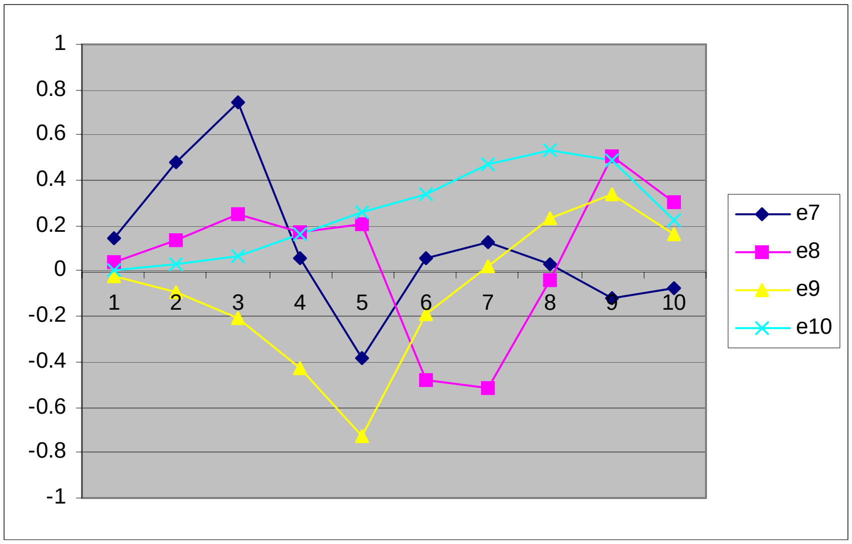


Figure 5. Eigenvectors associated with first four biggest values for optimized Λ_i

Although the eigenvectors do not have the typical humps presented in many books, these values generate very small differences between the theoretical and market swaption volatilities.

The biggest differences are denoted for 4 to 5 year length underlying swaps. The other differences for the volatilities in other maturities are not significant. This suggests that this type of calibration may be widely used in practice for valuation of various interest rate derivatives.

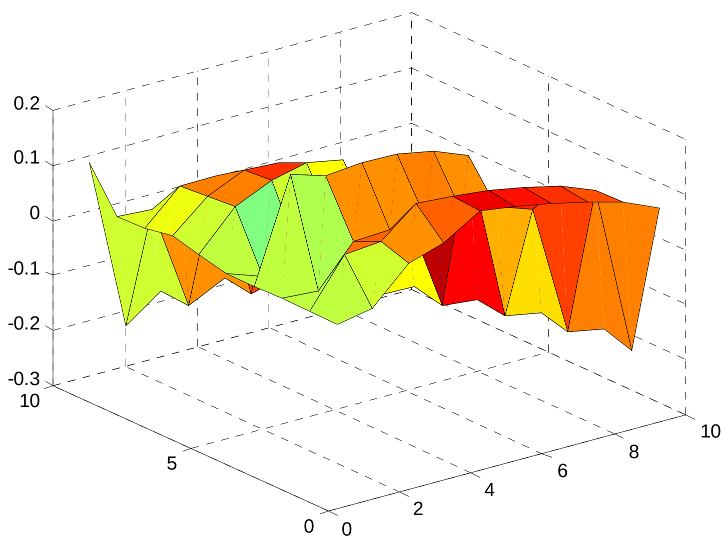


Figure 6. Algebraic difference between theoretical and market swaption volatilities

We have below a graphical representation of the optimized parameters Λ_i:

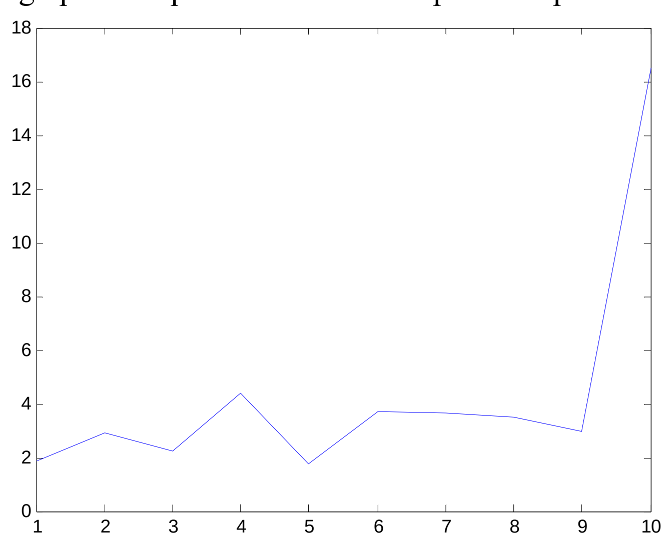


Figure 7. Parameters Λ_i obtained through optimization

The RSME (root mean squared error for differences between theoretical and market swaption volatilities) for 100 iterations is 0.47053.

If we increase the number of iterations in the optimization function, RSME has a much smaller value 0.013019, therefore the results obtained by means of calibration using the separated approach are much more accurate.

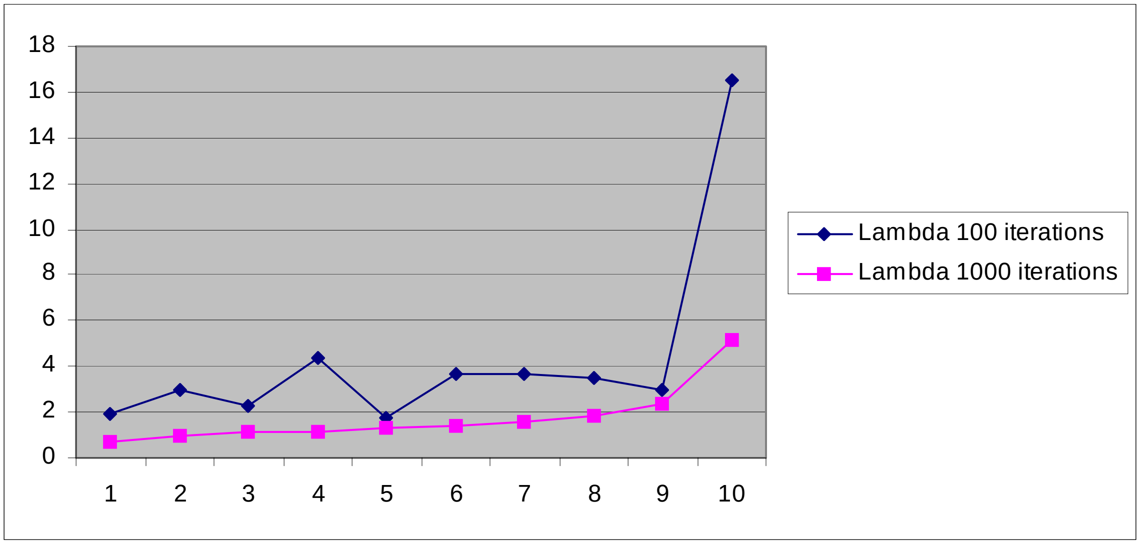


Figure 8. Parameters obtained for Λ_i through optimization with 1000 iterations as opposed to 100 iterations. ($RSME_{100} = 0.47053$; $RSME_{1000} = 0.013019$)

Theoretical volatilities 100 iterations — Theoretical volatilities 1000 iterations

$RSME_{100} = 0.47053$ — $RSME_{1000} = 0.013019$

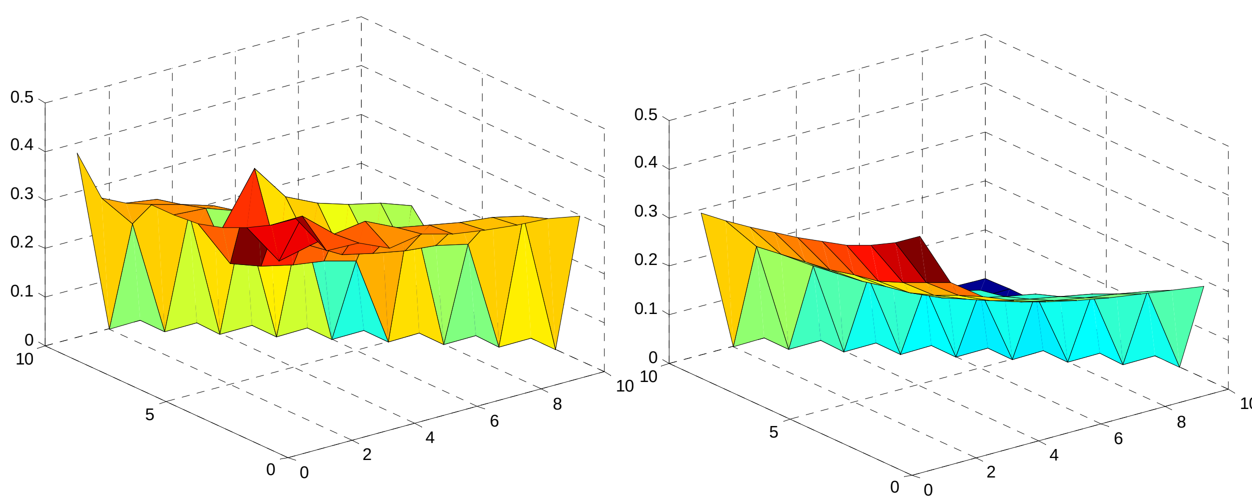


Figure 9. Theoretical volatilities computed from optimization with 100/1000 iterations

Algebraic difference between theoretical and market swaption volatilities
100 iterations
$RSME_{100} = 0.47053$

Algebraic difference between theoretical and market swaption volatilities
1000 iterations
$RSME_{1000} = 0.013019$

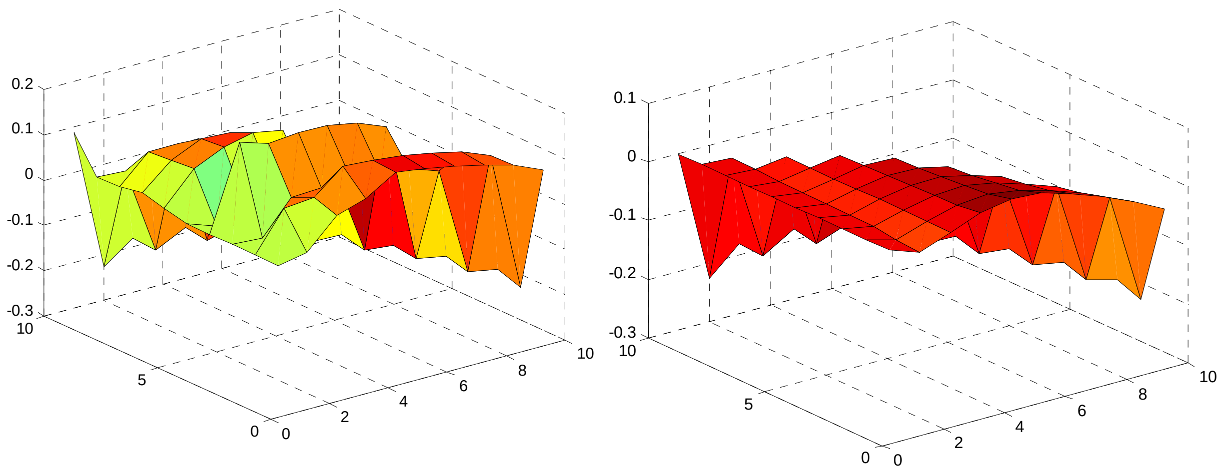


Figure 9. Algebraic difference between theoretical and market swaption volatilities computed from optimization with 100/1000 iterations

Therefore, in order to obtain a smaller RSME we need to increase the number of iterations.

In "LIBOR Market Model in Practice", Gatarek, Bachert and Maksymiuk show that the results obtained by means of optimization give a much better RSME levels than by choosing an arbitrary function for Λ_i such as:

- $\Lambda_i = e^{\delta i}$
- $\Lambda_i = (e^{\delta i})^{1/2}$
- $\Lambda_i = \delta_i - \ln(\delta_i)$
- $\Lambda_i = [\delta_i - \ln(\delta_i)]^{1/2}$
- $\Lambda_i = 1$

5.2 Results - Approximate Solutions for Calibration

Empirical Results on European Swaptions

It is indeed plausible that the instantaneous volatility of a swap rate might be evaluated with sufficient precision by calculating the stochastic coefficients ζ using the initial yield curve. The ultimate proof of the validity of the procedure, however, is obtained by checking actual European swaption prices.

The following test was therefore carried out.

- The instantaneous volatility function described in equation (A) was used, with parameters chosen as to ensure a realistic and approximately time homogeneous behaviour for the evolution of the term structure of volatilities: a =5%, b = 0.5, c = 1.5, and d = 15%. In particular, the values of the vector k defined by equation (A) were set to unity, thereby ensuring a time-homogeneous evolution of the term structure of volatilities;
- The correlation amongst the forward rates was assumed to be given by equation (B) with β= 0.1;
- Given this parametrised form for the forward-rate instantaneous volatility, the instantaneous volatility of a given swap was integrated out to the expiry of the chosen European swaption. (C). The value of this integral could therefore be evaluated analytically and gave the required approximate implied volatility for the chosen European swaption;
- With this implied volatility the corresponding approximate Black price was obtained;
- Given the initial yield curve and the chosen instantaneous volatility function for the forward rates, the European swaption prices were computed. For this evaluation, the same correlation function was used in the estimation of the approximate price;

Code in Matlab used for the calibration as well as calibration results can be found in Appendix.

The results are shown below.

If we compare the Black volatility estimated by using this calibration technique (approximate functions proposed by Rebonato for the instantaneous volatility and instantaneous correlation and the approximative coefficients ζ -using the initial yield curve) with the market Black quoted volatilities, we can see that the biggest differences are denoted for swaptions with long implied volatilities. Swaptions starting in 4 years as well as long end starting swaptions have theoretical implied Black volatilities higher than the market quoted swaptions.

Therefore, as parameters for the evolution of the term structure of volatilities were chosen to be: a =5%, b = 0.5, c = 1.5, and d = 15%, we can conclude that that the very-long maturities implied volatilities might not be accurately specified “d”.

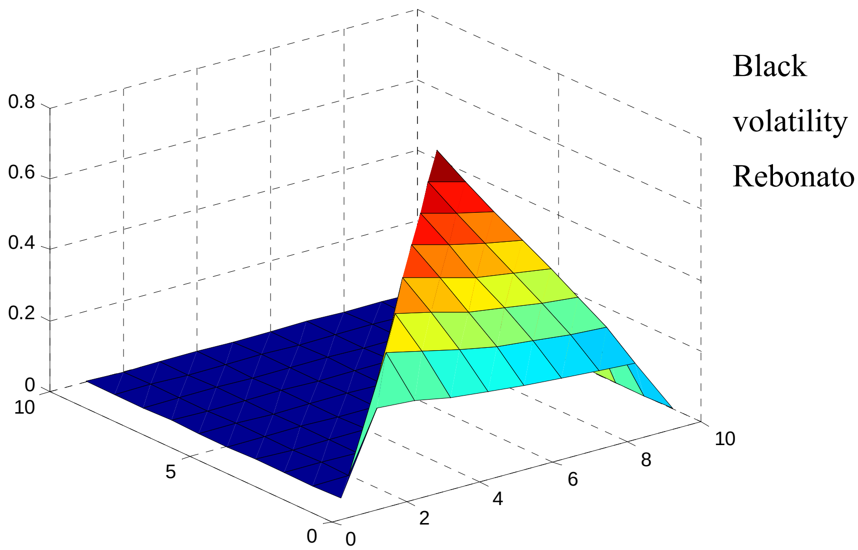
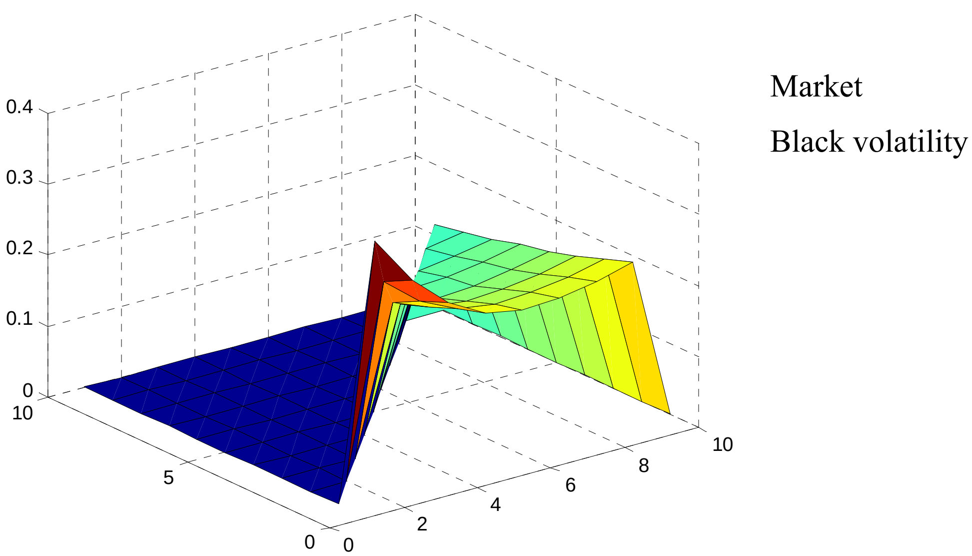


Figure 10. Black volatility computed with Rebonato’s approximate functions vs. market volatility



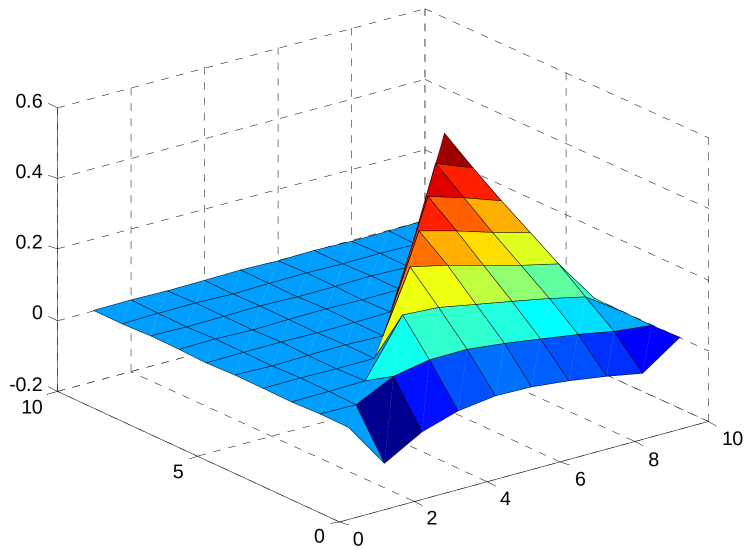


Figure 11. Algebraic difference between theoretical and market swaption volatilities computed from Rebonato's approximation

RSME (the squared mean root between theoretical and market volatilities) is 0.34032. The error is comparable with the one obtained from the previous calibration, however theoretical Black volatilities are differently distributed, the main difference coming from the long-end starting swaptions.

In Figure 12, you can observe the prices for the swaptions determined by using the approximate functions for volatility and correlation proposed by Rebonato.

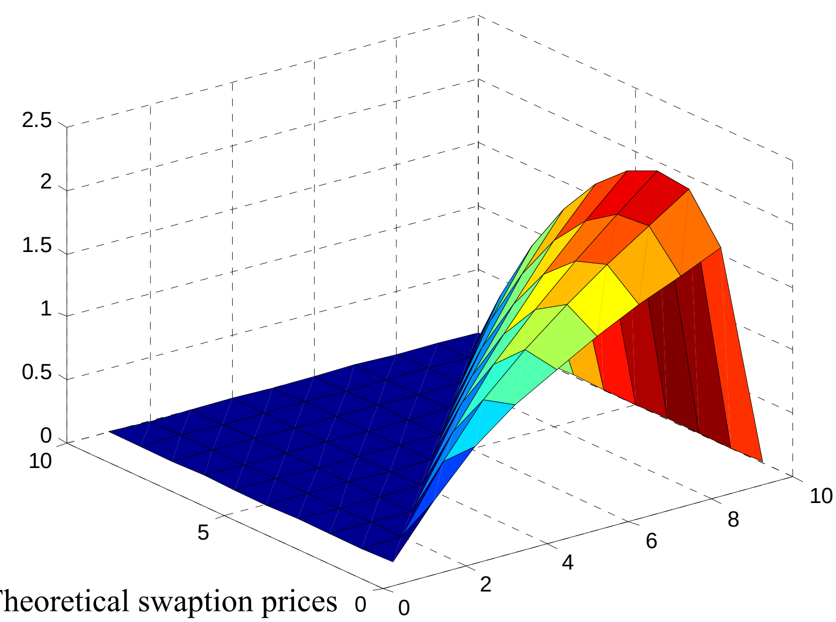
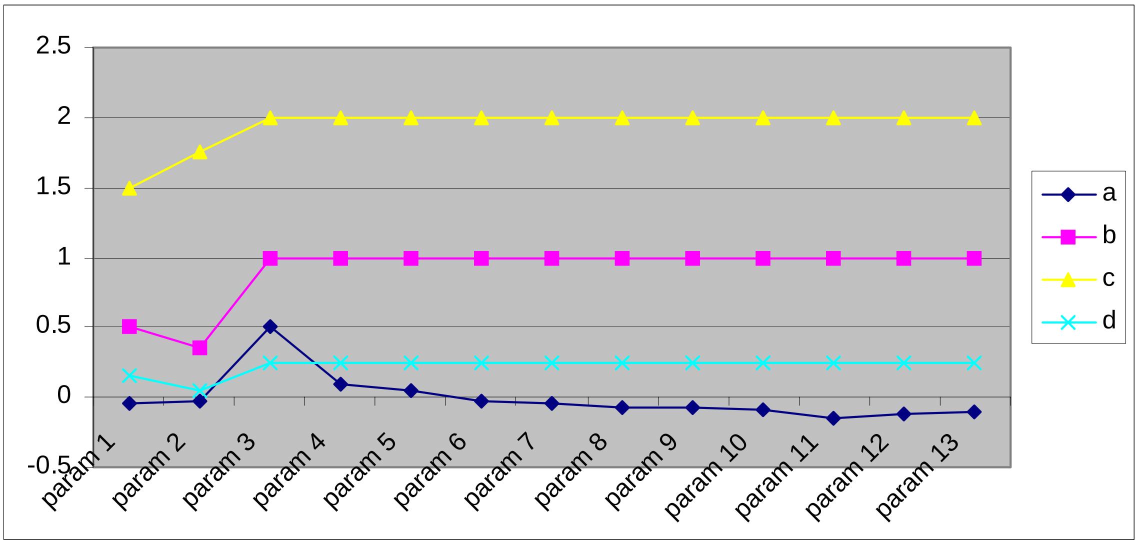
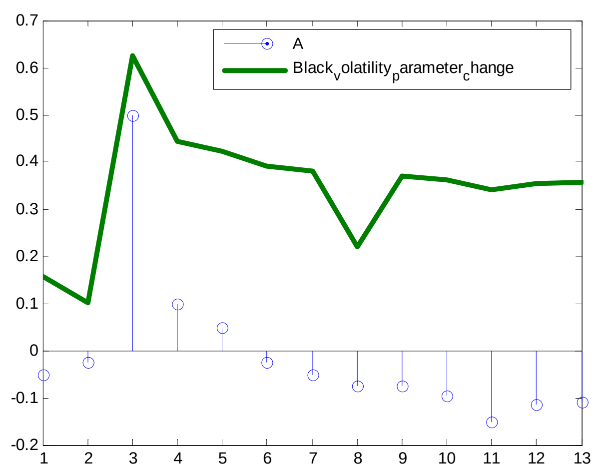


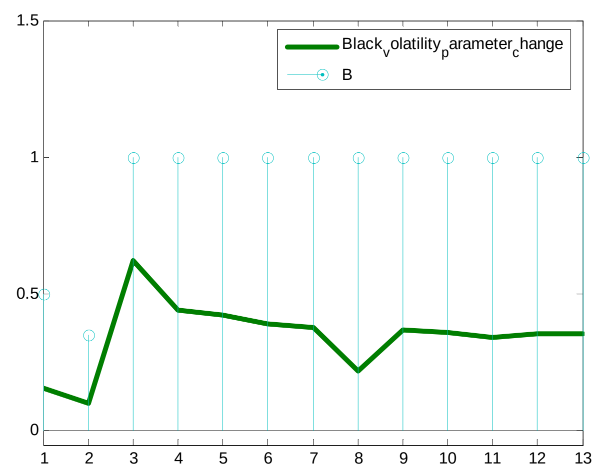
Figure 12: Theoretical swaption prices

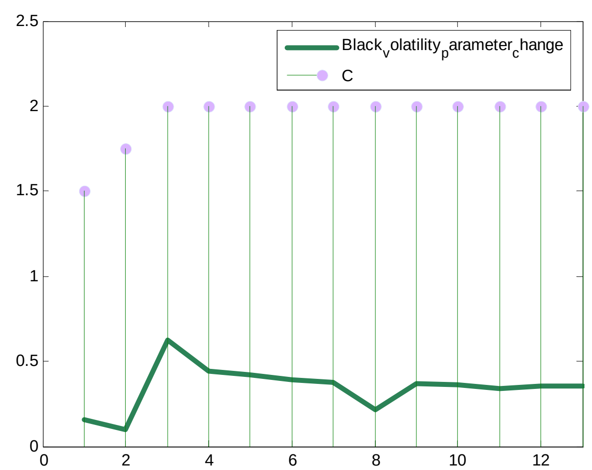
After computing the whole swaption matrix for the initially chosen parameters: a = -0.05; b = 0.5; c = 1.5; d = 0.15; I checked the impact of a change in parameters on a swaption price and on the theoretically quantified Black volatility. The exercise was done for the swaption with the underlying swap starting one year from today and maturing one year after (1,2 swaption).

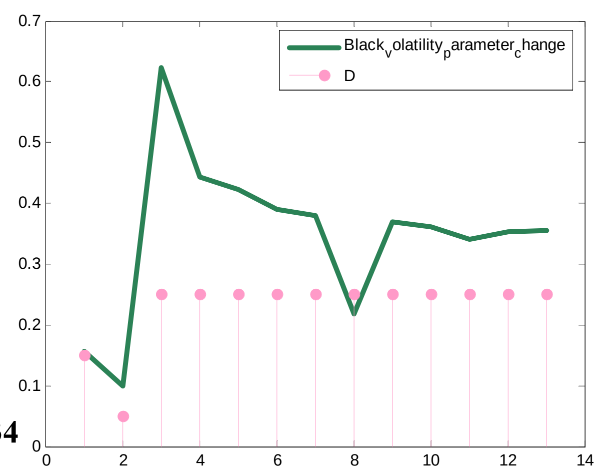
The choice of parameters can be found below (**Figure 13**)











By changing the initially chosen parameters: a, b, c, d from the instantaneous volatility function proposed by Rebonato, I obtained a theoretical Black volatility for the 1.2 swaption which is almost the same as the one quoted in the market. Parameter "a" has the greatest impact on the quantified volatility. Moreover, I increased the value of "d" (gives the very-long maturities implied volatilities), and the value of "c" (gives the sensitivity of the instantaneous volatility to the changes in the relative distance in years between two forward rates).

Therefore, we can conclude that the second method of calibration gives better results if we optimize the parameters used as inputs for the instanataneou volatility and correlation function $\{a, b, c, d, \beta\}$.

6. Concluding Remarks

The goal of this paper was to compare two methods of calibration for the LIBOR Market Model: the separated approach with optimization (Gaterek) and the calibration using the approximate solution proposed by Rebonato&Jackel.

As far as the separated approach is concerned, it is based on the assumption that $\varphi_{kl}^{i} = \Lambda_i * \varphi_{kl}$, where φ_{kl} is an element of the covariance matrix. The calibration using the separated approach with optimization minimizes the root mean squared error for the differences between theoretical and market swaption volatilities. Therefore the Lambda parameters are computed, in order to minimize the error. The conclusion drawn is that the method is accurate and provides good results for the error but is computationally intensive. If we increase the number of iterations in order to obtain a smaller error, even if the results do improve significantly, the computation also becomes quite lengthy.

As for the second technique of calibration, using the approximate solutions proposed by Rebonato&Jackel, suffices to price European swaptions with a remarkable degree of accuracy. The mechanism responsible for this surprisingly good match using an approximate equivalent volatility was explained in the theoretical part. The error between theoretical and market prices for swaptions is similar to the error obtained from the first calibration technique (with 100 iterations). However, this error can be further minimized by optimizing the input parameters of the instantaneous volatility and correlation function. The issue of how the parameters of the forward rate instantaneous volatility function should be chosen is assumed to have been separately resolved to the trader's liking. The trader could be interested in obtaining the best fit to the overall swaption matrix while pricing the caplets exactly. In order to achieve this task, one can first optimize iteratively over the parameters so as to find the set of {a, b, c, d, β} that best accounts for the swaption matrix.

By using the approximation for ζ the trader can tell in a quick and accurate way how well the swaption matrix is reproduced by the chosen functional form for the forward-rate instantaneous volatilities and correlations. A full swaption matrix can therefore be obtained in under a second.

The main conclusion is that both methods of calibration offer significant results. The techniques can be improved in order to minimize the error of calibration as follows:

- the separated approach with optimization (Gaterek) - increase the number of iteration with the disadvantage of a long lasting computation
- the calibration using the approximate solution proposed by Rebonato&Jackel – optimize the value of the input parameters {a, b, c, d, β} of the instantaneous volatility and correlation functions in order to better fit the quoted swaption volatility matrix.

References

[1] Alexander C. (2002), "Common Correlation Structures for Calibrating the LIBOR Model", ISMA Centre Finance Discussion Paper No. 2002-18

[2] Brace A., Gatarek D. and Musiela M. (1997), "The market model of interest rate dynamics", Mathematical Finance, 127–155

[3] De Jong F., Driessen J., Pelsser A. (2001), "Libor Market Models versus Swap Market Models for Pricing Interest Rate Derivatives -An Empirical Analysis", European Finance Review,201-237

[4] Gatarek D.,Bachert P. and Maksymiuk R (2006),"The LIBOR Market Model in Practice", John Wiley and Sons, Ltd, 1-21, 27-37, 63-167

[5] Jackel P. (2002), "Monte Carlo Methods in Finance", John Wiley and Sons

[6] Jackel P. and Rebonato R.(2001) "Linking Caplet and Swaption Volatilities in a BGM/J Framework: Approximate Solutions", Journal of Computational Finance

[7] Kajsajuntti L.(2004), "Pricing of Interest Rate Derivatives with the LIBOR Market Model", Royal Institute of Technology Stockholm

[8] Rebonato R.(2002), "Modern Pricing of Interest-rate Derivatives. The LIBOR Market Model and Beyond", Princeton University Press, Princeton and Oxford, 3-57, 135-209, 276-331

[9] Rebonato R.(1999)"On the simultaneous calibration of multi-factor log-normal interest-rate models to Black volatilities and to the correlation matrix", Journal of Computational Finance

[10] Vojteky M.(2004), "Calibration of Interest Rate Models - Transition Market Case" , CERGE-EI Working Paper No. 23

APPENDIX

1. Results of computations - Calibration in the separated approach with optimization

RSME (Root Mean squared error between theoretical and market swaption volatilities)

0.47503

Lambda

1.8945
2.9466
2.2588
4.3705
1.7558
3.709
3.6618
3.4925
2.9476
16.478

L (vector of eigenvalues)

-0.077352
-0.037983
0.0076164
0.060573
0.084643
0.11274
0.24182
0.28975
0.46193
0.63266

Matrix of eigenvectors

-0.0005025	-0.0006838	0.02485	0.44598	-0.034679	0.88069	0.14807	0.0359	-0.019707	0.0079537
-0.0019767	-0.0013833	-0.1083	-0.79336	0.01709	0.31697	0.47905	0.13612	-0.094556	0.033454
0.0027443	-0.0034131	0.33769	0.34393	0.0044404	-0.32397	0.7441	0.24696	-0.20873	0.066659
-0.020578	0.014617	-0.84251	0.19429	-0.035967	-0.10362	0.057411	0.17026	-0.42471	0.1646
-0.026676	0.051016	0.403	-0.12388	-0.14549	0.083271	-0.38549	0.20986	-0.7297	0.25637
0.27543	-0.40154	0.030252	0.0085615	0.6168	0.021125	0.058734	-0.47642	-0.19328	0.33651
-0.48266	0.5032	0.0098406	0.0006262	-0.11348	-0.0097982	0.13193	-0.51429	0.026167	0.46709
0.49602	-0.21389	-0.011927	-0.012291	-0.60716	-0.020975	0.034935	-0.03921	0.23719	0.52898
-0.46337	-0.30857	0.014613	0.0028677	0.24829	0.010416	-0.12262	0.50889	0.33786	0.48984
0.47883	0.66465	0.0029016	0.0078838	0.3904	0.014056	-0.076786	0.30691	0.16067	0.22444

VCV (matrix of covariances)

0.10549	0.029568	0.0086234	0.0034274	0.0017377	0.0011244	0.0008598	0.0007246	0.0006519	0.0006088
0.029568	0.11527	0.07809	0.023001	0.0093028	0.0048098	0.003143	0.0023735	0.0019713	0.0017358
0.0086234	0.07809	0.19437	0.076062	0.022192	0.008909	0.0046347	0.0030371	0.0023022	0.0019081
0.0034274	0.023001	0.076062	0.11863	0.17021	0.048717	0.019325	0.010037	0.0066259	0.0050401
0.0017377	0.0093028	0.022192	0.17021	0.34082	0.07926	0.022729	0.0089789	0.0046458	0.0030389
0.0011244	0.0048098	0.008909	0.048717	0.07926	0.17577	0.18199	0.051763	0.02029	0.010337
0.00085983	0.003143	0.0046347	0.019325	0.022729	0.18199	0.19266	0.1946	0.055296	0.021491
0.00072457	0.0023735	0.0030371	0.010037	0.0089789	0.051763	0.1946	0.21425	0.19662	0.055505
0.00065187	0.0019713	0.0023022	0.0066259	0.0046458	0.02029	0.055296	0.19662	0.26821	0.17534
0.00060881	0.0017358	0.0019081	0.0050401	0.0030389	0.010337	0.021491	0.055505	0.17534	0.050924

VCV-M (matrix of modified covariances)

0.10549	0.029568	0.0086234	0.0034278	0.0017374	0.0011241	0.0008655	0.0007109	0.0006779	0.0005729
0.029568	0.11527	0.07809	0.023003	0.0093042	0.0047888	0.0031904	0.0023089	0.0020583	0.0016276
0.0086234	0.07809	0.19437	0.076056	0.022179	0.0090196	0.004467	0.0031701	0.0022438	0.0019236
0.0034278	0.023003	0.076056	0.11867	0.17028	0.048056	0.020373	0.0091282	0.0071922	0.0046469
0.0017374	0.0093042	0.022179	0.17028	0.34097	0.077914	0.0247	0.0075409	0.005004	0.0033388
0.0011241	0.0047888	0.0090196	0.048056	0.077914	0.18776	0.16403	0.065593	0.015124	0.010401
0.00086552	0.0031904	0.004467	0.020373	0.0247	0.16403	0.22029	0.172	0.066698	0.016318
0.00071085	0.0023089	0.0031701	0.0091282	0.0075409	0.065593	0.172	0.23502	0.18134	0.068477
0.0006779	0.0020583	0.0022438	0.0071922	0.005004	0.015124	0.066698	0.18134	0.28844	0.15038
0.00057294	0.0016276	0.0019236	0.0046469	0.0033388	0.010401	0.016318	0.068477	0.15038	0.085439

Sig-theo (Theoretical swaption volatilities)

0.45015	0.36624	0.36472	0.33614	0.34875	0.32953	0.32092	0.31473	0.30977	0.29848
0.41495	0.4139	0.36014	0.36947	0.33969	0.32532	0.31505	0.30697	0.29284	
0.3852	0.29983	0.3141	0.27888	0.26273	0.25148	0.24281	0.22922		
0.3627	0.46749	0.3908	0.36024	0.33978	0.32446	0.3024			
0.34853	0.24806	0.22558	0.21119	0.20015	0.18405				
0.34312	0.33945	0.31707	0.2969	0.26755					
0.34189	0.32552	0.30508	0.26751						
0.32263	0.31292	0.26473							
0.30957	0.2382								
0.37791									

Sig (market swaption volatilities)

0.45015	0.35627	0.2887	0.24184	0.21346	0.20411	0.20632	0.21522	0.22867	0.24539
0.41495	0.3282	0.26669	0.22478	0.20061	0.19324	0.19517	0.20272	0.21415	
0.3852	0.30195	0.24389	0.20515	0.18411	0.17849	0.18108	0.18874		
0.36264	0.28121	0.22529	0.18872	0.16993	0.16566	0.16898			
0.34845	0.27074	0.21726	0.18214	0.16384	0.15943				
0.33199	0.25764	0.20625	0.17227	0.15407					
0.31972	0.24824	0.19853	0.16535						
0.30805	0.23903	0.19076							
0.29852	0.23166								
0.29176									

2. Results of computations - Calibration by means of Approximate solutions according to Rebonato and Jackel

RSME 0.34032

Black volatility theoretical

0	0.22574	0.21901	0.20204	0.18793	0.17727	0.16903	0.16244	0.15697
0	0	0.31875	0.2955	0.27135	0.25317	0.23975	0.2295	0.22129
0	0	0	0.36591	0.3386	0.31321	0.29443	0.28065	0.27003
0	0	0	0	0.39915	0.37117	0.34603	0.32759	0.31393
0	0	0	0	0	0.42738	0.39953	0.37504	0.35699
0	0	0	0	0	0	0.45319	0.42569	0.40175
0	0	0	0	0	0	0	0.47743	0.45018
0	0	0	0	0	0	0	0	0.50046
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Black volatility market

0	0.35627	0.2887	0.24184	0.21346	0.20411	0.20632	0.21522	0.22867
0	0	0.26669	0.22478	0.20061	0.19324	0.19517	0.20272	0.21415
0	0	0	0.20515	0.18411	0.17849	0.18108	0.18874	0.19987
0	0	0	0	0.16993	0.16566	0.16898	0.177	0.18823
0	0	0	0	0	0.15943	0.1623	0.16968	0.18014
0	0	0	0	0	0	0.1523	0.15945	0.1696
0	0	0	0	0	0	0	0.15117	0.16086
0	0	0	0	0	0	0	0	0.1524
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Algaebraic Differences

0	-0.13053	-0.06969	-0.039801	-0.025523	-0.026842	-0.037291	-0.052779	-0.071701
0	0	0.05206	0.070727	0.070739	0.059927	0.044586	0.02678	0.0071394
0	0	0	0.16076	0.1545	0.13472	0.11335	0.091917	0.070166
0	0	0	0	0.22922	0.20551	0.17706	0.15059	0.1257
0	0	0	0	0	0.26795	0.23723	0.20536	0.17685
0	0	0	0	0	0	0.30089	0.26624	0.23214
0	0	0	0	0	0	0	0.32627	0.28932
0	0	0	0	0	0	0	0	0.34806
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Swaption Price Matrix

0	0.39678	0.73559	0.97891	1.1757	1.3492	1.5082	1.655	1.7911
0	0	0.5091	0.91035	1.2183	1.4795	1.7153	1.9323	2.1334
0	0	0	0.54143	0.97719	1.3275	1.6334	1.9118	2.169
0	0	0	0	0.5598	1.0215	1.4048	1.7436	2.0537
0	0	0	0	0	0.57514	1.0589	1.467	1.8312
0	0	0	0	0	0	0.58958	1.0894	1.5168
0	0	0	0	0	0	0	0.5989	1.111
0	0	0	0	0	0	0	0	0.60545
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

3. Matlab Routine for Calibration in the separated approach with optimization

%Step1: The Matrix of Parameters [R]
%Input: Vector of Discount Factors [B]
%Output: Matrix of parameters [R]

```
m=10; % Number of swaption maturities
M=20; % Number of swaption maturities + number of swaption underlyings
R=[]; % setting zeros  for matrix R as initial values
for i=1:m
for j=i+1:M-m+i
        for k=i+1:j
        R(i,j,k)=(B(k-1)-B(k))/(B(i)-B(j));
end
end
end
```

%Step2: The matrix of covariances [VCV] as a function of parameters
%Input: (1) Matrix of parameters [R]
%   (2) Vector of Dates [T_num]
%   (3) Matrix of market swaption volatilities [Sig]
%   (4) Vector of initial parameters [Lambda]
%Output: Matrix of covarinaces (VCV) as a function of parameters (Lambda)

```
VCV=[]; % setting zeros for matrix VCV as initial values
%Diagonal elements of matrix VCV
for k=1:m
        VCV(k,k)=yearfrac(Today,T_Num(k))*Sig(k,1)^2/Lambda(k);
end
s=1;
for i=1:m
        for j=i+1:m
        Sum=0;
for l=i+j-2*s+1:j+1
        for k=i+j-2*s+1:j+1
                SumTemp=R(i+j-2*s,j+1,k)*R(i+j-2*s,j+1,1)*VCV(k-1,l-1);
                Sum=Sum+SumTemp;
        end
end
VCV(i+j-2*s,j)=((yearfrac(Today,T_Num(i+j-2*s))*Sig(i+j-2*s,i+1)^2-Lambda(i+j-
2*s)*(Sum-2*R(i+j-2*s,j+1,                    i+j-2*s+1)*VCV(i+j-2*s,j)*R(i+j-
2*s,j+1,j+1)))/2*Lambda(i+j-2*s)*R(i+j-2*s,j+1, i+j-2*s+1)*R(i+j-2*s,j+1,j+1));
```

```
VCV(j, i+j-2*s)=VCV(i+j-2*s,j);
end
s=s+1
end
```

%Step 3: The vector of EigenValues [L] and the parix of eigenvestors [E] as a %function of parameters [Lambda]

```
%Input: Matrix of covariances [VCV] as a function of parameters [Lambda]
%Output: (1) Vector of Eigenvalues [L] as a function of parameters [Lambda]
        %(2) Matrix of eigenvectors [E] as a function of parameters [Lambda]

[E,X]=eig(VCV);
L=diag(X);
```

%Step 4: The modified covariance matrix [VCV_M] as a function of parameters %lambda

```
%Input: (1) Vector of Eigenvalues [L] as a function of parameters [Lambda]
       %(2) Matrix of eigenvectors [E] as a function of parameters [Lambda]
%Output: Modified covariance matrix [VCV_M] as a function of parameters [Lambda]

% Step 4 contains a sub-algorithm for eliminating eigenvectors associated with negative
%eigenvalues

for i=1:m
if L(i)<0
        L_check(i)=1;
else
        L_check(i)=0;
end
end
```

%Matrix [E_sqrL] constructed by multiplying eigenvectors by square root of %associated positive eigenvalues

```
for i =1:m
        if L_check(i)==0
for j=1:m
        E_sqrL(j,i)=E(j,i)*sqrt(L(i));
end
else
for j=1:m
        E_sqrL(j,i)=0
end
end
end
VCV_M= E_sqrL* transpose(E_sqrL); % transposed
```

%Step 5: Calculation of theoretical swaption volatilities [Sig_theo]

```
%Input: (1) Matrix of parameters [R]
          %  (2): Matrix of modified covariance [VCV_M]
%Output: Matrix of theoretical swaption volatilities [Sig_theo]



Sig_theo=[];
for k=1:m
for N=k+1:m+1
Sum=0;
for l=k+1:N
for i=k+1:N
SumTemp=R(k,N,i)*VCV_M(i-1,l-1)*R(k,N,l);
Sum=Sum+SumTemp;
end
end
Sig_theo(k,N-k)=sqrt(Sum*Lambda(k)/yearfrac(Today,T_Num(k)));
end
end
```

%Step 6: RSME between theoretical and market swaption volatilities

```
%Input: (1) Matrix of theoretical colatilities [Sig_theo]
          % (2) Matrix of market swaption volatilities [Sig]
%Output: RSME between th and market swaption volatilities

RSME=0
for i=1:m
for j=1:m-i+1
RSME_Temp=(Sig_theo(i,j)-Sig(i,j))^2;
RSME=RSME+RSME_Temp;
end
end
f=RSME;

% function f will be used as a minimization function


options=optimset('MaxIter',100)
```

[Lambda,f]=fminsearch(@CalibrationObjectiveFunction_SeparatedOptim,Lambda0,options);

4. Matlab Routine for Calibration by means of Approximate solutions according to Rebonato and Jackel

Functions used:

```
%Based on eq 2
function ret= W0fn(i)
 global P;
 global tau;
 global alpha;
 global beta;
 tmpsum=0;
 for i=alpha:beta-1,
  tmpsum=tmpsum+P(i+1)*tau;
 end
 ret=(P(i+1)*tau)/tmpsum;

for i=alpha:beta-1
W(i)= W0fn(i);
end


%based on eq 6
function ret= eta0_approx(j,k)
 global F;
 global alpha;
 global beta;
 tmpsum=0;
 for i=alpha:beta-1,
  tmpsum=tmpsum + W0fn(i)*F(i);
 end
 ret= (W0fn(j)*F(j)*W0fn(k)*F(k)) / (tmpsum^2);


%returns fair rate of swap
function ret = GetSwapRate(F,alpha,beta)
        global tau;
        tmp_sum=0;
        SR=1;
        tmp=1;
        for j=alpha:beta-1,
                tmp=tmp*(1/(1+tau*F(j)));
```

```
        end
        SR=1-tmp; %numerator
        for i=alpha:beta-1,
                tmp=1;
                for j=alpha:i,
                        tmp=tmp*(1/(1+tau*F(j)));
                end
                tmp_sum=tmp_sum + (tau*tmp);
        end
        SR=SR/tmp_sum;
        ret=SR;
```

%From analytical formula

```
function ret= IndefinteIntegral(i,j,t)
 global a;
 global b;
 global c;
 global d;
 global T;
 vol_beta=0.1;
 ti=T(i);
 tj=T(j);
 tmp=4*a*c^2*d*  (  exp(c*(t-tj))+exp(c*(t-ti))  );
 tmp=tmp+ 4*c^3*d^2*t;
 tmp=tmp - 4*b*c*d*exp(c*(t-ti))  *  (c*(t-ti)-1);
 tmp=tmp - 4*b*c*d*exp(c*(t-tj))  *  (c*(t-tj)-1);
 tmp2=2*a^2*c^2;
 tmp2=tmp2+ 2*a*b*c*(1+(ti+tj-2*t));
 tmp2=tmp2+ b^2*(1 + 2*c^2*(t-ti)*(t-tj) + c*(ti+tj-2*t)  );
 tmp=tmp+exp(c*(2*t-ti-tj))*tmp2;
 ret=exp(-vol_beta*abs(ti-tj))   *    (1/(4*c^3))  * tmp;
```

%based on eq 12

```
function ret= Afn(i)
 global P;
 global F;
 global tau;
 global beta;
 tmpsum=0;
 for j=i:beta-1,
  tmpsum=tmpsum+ P(j+1)*F(j)*tau;
 end
 ret=tmpsum;
```

```
%based on eq 13
function ret= Bfn(i)
 global P;
 global tau;
 global beta;
 tmpsum=0;
 for j=i:beta-1,
  tmpsum=tmpsum+ P(j+1)*tau;
 end
 ret=tmpsum;


%revised formula for etaij as mentioned in eq 15
function ret= eta0(i,j)
 global P;
 global tau;
 global F;
 global alpha;
 global beta;
 tmp1= (P(i+1)*F(i)*tau)/ (Afn(alpha));
 tmp11= (Afn(alpha)*Bfn(i)-Afn(i)*Bfn(alpha))*F(i)*tau;
 tmp11=tmp11/ (Afn(alpha)*Bfn(alpha)*(1+F(i)*tau));
 tmp1=tmp1 +tmp11;
 tmp2= (P(j+1)*F(j)*tau)/ (Afn(alpha));
 tmp22= (Afn(alpha)*Bfn(j)-Afn(j)*Bfn(alpha))*F(j)*tau;
 tmp22=tmp22/ (Afn(alpha)*Bfn(alpha)*(1+F(j)*tau));
 tmp2=tmp2 +tmp22;
 ret=tmp1*tmp2;


%black price
function ret= Black(K,Forward,v)
 d1=(log(Forward / K) + 0.5 * v * v) / v;
 d2 = d1 - v;
 Nd1=normal_cdf(d1);
 Nd2=normal_cdf(d2);
 ret=(Forward * Nd1 - K * Nd2);
```

Matlab routine:

```
global tau;
global P;
global T;
global F;
global a;
global b;
global c;
global d;
global alpha;
global beta;


tau = 1; %indexing interval
alpha=10; %start peg of swap
beta=10; %end peg of swap

a = -0.05;
b = 0.5;
c = 1.5;
d = 0.15;

P=B;
%set to appropriate discount curve to use
F=zeros(size(P,1)-1,1); %forward rate curve
T=0:tau:10;
for i=1:size(F),
        F(i)=(P(i)/P(i+1)-1)/tau;
end


tmpsum=0;
for j=alpha:beta-1,
 for k=alpha:beta-1,
  tmp=IndefinteIntegral(j,k,T(alpha))-IndefinteIntegral(j,k,0);
  tmpsum=tmpsum+ eta0(j,k)*tmp;   %use eta0_approx for eq 6
 end
end

black_volatity=tmpsum^0.5;
tmpsum=0;
for i=alpha:beta-1,
 tmpsum=tmpsum+tau*P(i+1);
end
```

```
SR=GetSwapRate(F,alpha,beta); %fair value of swap
K=SR; %swaption is ATM

swaption_price=tmpsum*Black(K,K,black_volatity)*100;
tmpblack=Black(K,K,black_volatity);
tmpstr=sprintf('swap start=%f   \nswap end=%f     \nswaption
price=%f',T(alpha),T(beta),swaption_price);
disp(tmpstr);

Black_Vol_Matrix =[0 black_volatity_1_2 black_volatity_1_3 black_volatity_1_4
black_volatity_1_5 black_volatity_1_6 black_volatity_1_7 black_volatity_1_8
black_volatity_1_9 0; 0 0 black_volatity_2_3 black_volatity_2_4 black_volatity_2_5
black_volatity_2_6 black_volatity_2_7 black_volatity_2_8 black_volatity_2_9 0; 0 0 0
black_volatity_3_4 black_volatity_3_5 black_volatity_3_6 black_volatity_3_7
black_volatity_3_8 black_volatity_3_9 0; 0 0 0 0 black_volatity_4_5 black_volatity_4_6
black_volatity_4_7 black_volatity_4_8 black_volatity_4_9 0; 0 0 0 0 0
black_volatity_5_6 black_volatity_5_7 black_volatity_5_8 black_volatity_5_9 0; 0 0 0 0
0 0 black_volatity_6_7 black_volatity_6_8 black_volatity_6_9 0; 0 0 0 0 0 0 0
black_volatity_7_8 black_volatity_7_9 0; 0 0 0 0 0 0 0 0 black_volatity_8_9 0; 0 0 0 0 0
0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0];

Swaption_Price_Matrix =[0 swaption_price_1_2 swaption_price_1_3
swaption_price_1_4 swaption_price_1_5 swaption_price_1_6 swaption_price_1_7
swaption_price_1_8 swaption_price_1_9 0; 0 0 swaption_price_2_3 swaption_price_2_4
swaption_price_2_5 swaption_price_2_6 swaption_price_2_7 swaption_price_2_8
swaption_price_2_9 0; 0 0 0 swaption_price_3_4 swaption_price_3_5
swaption_price_3_6 swaption_price_3_7 swaption_price_3_8 swaption_price_3_9 0; 0 0
0 0 swaption_price_4_5 swaption_price_4_6 swaption_price_4_7 swaption_price_4_8
swaption_price_4_9 0; 0 0 0 0 0 swaption_price_5_6 swaption_price_5_7
swaption_price_5_8 swaption_price_5_9 0; 0 0 0 0 0 0 swaption_price_6_7
swaption_price_6_8 swaption_price_6_9 0; 0 0 0 0 0 0 0 swaption_price_7_8
swaption_price_7_9 0; 0 0 0 0 0 0 0 0 swaption_price_8_9 0; 0 0 0 0 0 0 0 0 0 0; 0 0 0 0
0 0 0 0 0 0]

m=10
RSME=0
for i=1:m
for j=1:m-i+1
RSME_Temp=(Vol_Market_Matrix_modified(i,j)- Black_Vol_Matrix (i,j))^2;
RSME=RSME+RSME_Temp;
end
end
```