Return to RiskMetrics: The Evolution of a Standard

Jorge Mina and Jerry Yi Xiao

Introduction by Christopher C. Finger

RiskMetrics
www.riskmetrics.com
44 Wall St.
New York, NY 10005

Foreword

This document is an update and restatement of the mathematical models in the 1996 *RiskMetrics Technical Document*, now known as *RiskMetrics Classic*. *RiskMetrics Classic* was the fourth edition, with the original document having been published in 1994. Since the initial publication, the model has become the standard in the field and is used extensively in practice, in academic studies, and as an educational tool. At the same time, the aim that risk models be transparent has become a guiding principle of the RiskMetrics Group, Inc. and has carried over to our subsequent models for credit, pension funds, and retail investors. However, there have been numerous modeling and technological advances, and the standard risk model has evolved significantly since 1996. While we at RiskMetrics have incorporated this evolution into our software offering and have regularly published updates to our methodology, it has been almost five years since we updated the formal statement of the model. Given our continued commitment to transparency, we have thus created this new document, *Return to RiskMetrics: The Evolution of a Standard*. We encourage our readers to provide feedback or submit questions by email at `rtr@riskmetrics.com`

The authors would like to thank Christopher Finger and Allan Malz of RiskMetrics for their continuous support and invaluable comments. We would also like to thank Anne and Arnie Sternheim for editing this document.

Contents

Introduction

In October 1994, the risk management group at J.P. Morgan took the bold step of revealing its internal risk management methodology through a fifty page technical document and a free data set providing volatility and correlation information for roughly twenty markets. At the time, there was little standardization in the marketplace, and the RiskMetrics model took hold as the benchmark for measuring financial risk. In the subsequent years, as the model became a standard for educating the industry as well, the demands for enhancements and advice grew. We continued to develop the model, and by mid-1998, the *Technical Document* had been updated three times, with the last release (the fourth edition, or *RiskMetrics Classic*) tipping the scales at almost 300 pages, more timely updates and advances had come in the form of thirteen *RiskMetrics Monitors*, and the free dataset had expanded to cover foreign exchange, equity, fixed income, and commodities in 33 countries. Demand for a straightforward implementation of the model arose as well, leading to the development of our first software product, *FourFifteen*.

In 1998, as client demand for the group's risk management expertise far exceeded the firm's internal risk management resources, RiskMetrics was spun off from J.P. Morgan. We have continued in our commitment to transparency, and have continued to publish enhancements to the RiskMetrics methodology, most recently in two issues of the *RiskMetrics Journal* in 2000. In total, we have now distributed approximately 100,000 physical copies of the various versions of the *Technical Document*, and still consistently provide over 1,000 electronic versions each month through our website. Meanwhile, the RiskMetrics datasets are still downloaded over 6,000 times each month.

Clearly, standards do not remain static as theoretical and technological advances allow for techniques that were unpractical or unknown previously and as new markets and financial products require new data sources and methods. We have faced these issues; the methodology employed in our second and third generation market risk applications represents a significant enhancement of the RiskMetrics model as documented in *RiskMetrics Classic*. Additionally, our experience, and the experience of the industry as a whole, has taught that a single risk statistic derived from a single model is inadequate, and as such, we have emphasized the use of alternative risk measures and stress tests in our software. So, while our model has evolved, and now represents a standard for the year 2001, the basic documentation still represents a standard for the year 1996, and a good deal has changed since then.

Looking back, we can divide the material covered in *RiskMetrics Classic* into three major pieces. The first of these, covered in Part One, contains the applications of the measures, or the "why" of risk measurement. In this area, regulatory standards have changed, as have disclosure and management practices. To address these changes, and to provide insight into risk management practices without delving into modeling details, we published *Risk Management: A Practical Guide* in 1999.

A second area, covered in Part Four of *RiskMetrics Classic*, concerns the market data that serves as the key input to the model. As we have covered more and broader markets, the data aspect of RiskMetrics has perhaps expanded more than any other area. We have formed a separate data service, DataMetrics, which now warehouses close to 100,000 series. Acknowledging the critical nature of this service, and its status as a product in itself, we will soon publish the *DataMetrics Technical Document*. This document covers market data sources used by DataMetrics, methods used to enhance the quality of the data, such as outlier identification, fitting of missing data, and synchronization, and analytics employed for derived data such as bootstrapped yield curves.

The third area, covered in Parts Two and Three of *RiskMetrics Classic*, is the mathematical assumptions used in the model itself. Although we have made significant enhancements to the models as represented in our software, our documentation has lagged this innovation and, unfortunately, *RiskMetrics Classic*, as a representation of our software, is slightly underwhelming. In other words, a self-contained statement of the standard risk model does not exist today. The first goal of *Return to RiskMetrics*, then, is to rectify this problem by documenting the updated market-standard risk methodology that we have actually already implemented.

As well as this update, we have seen the need to clarify a number of misconceptions that have arisen as a result of the acceptance of *RiskMetrics Classic*. Practitioners have come to equate Value-at-Risk (VaR), the variance-covariance method, and RiskMetrics. Thus, it is common that pundits will criticize RiskMetrics by demonstrating that VaR is not an appropriate measure of risk. This is really a criticism of the use of a percentile to measure risk, but not a criticism of the model used to compute the measure. At the same time, we hear critics of VaR who claim the method is deficient because it captures only linear positions. This is not a criticism of the risk measure, but rather of the classic RiskMetrics variance-covariance method used to compute the measure. To be clear, we state that VaR is not RiskMetrics, and, in fact, is a risk measure that could even be an output of a model at odds with our assumptions. By the same token, RiskMetrics is not VaR, but rather a model that can be used to calculate a variety of risk measures. Finally, RiskMetrics is not a single set of computational techniques and approximations, such as the linear portfolio assumption or the Monte Carlo procedure. Rather, RiskMetrics encompasses all of these within a hierarchy of solution techniques for the fundamental model.

A final goal to this exercise is one of introspection. We have spoken of clarifying what RiskMetrics is not; there also lies the more difficult task of illuminating what RiskMetrics is. In a very strict sense, RiskMetrics is two fundamental and battle-tested modeling assumptions: that returns on risk factors are normally distributed and that volatilities of risk factors are best estimated using an exponentially weighted moving average of past returns. These two assumptions carry over from *RiskMetrics Classic*. Since the volatility estimation procedure has not changed, and since its explanation in *RiskMetrics Classic* is clear, we will not repeat the discussion in this document. On the other hand, though the normality assumption has not changed, we have seen the need to present it differently for clarity. In Chapter 2, we state the assumptions more technically and discuss two frameworks to calculate risk measures within the model: the closed-form approach, which is simpler but requires more approximations, and the Monte Carlo approach, which is more exact but also more burdensome. Of course, two assumptions do not make a risk model, and even with these assumptions stated, the model is not complete. For instance, it is still necessary to specify the risk factors, to which we have devoted Chapter 1, and the instrument pricing functions, to which we have devoted Chapter 5.

More generally, a risk model does not make a risk management practice. This brings us to a broader definition of RiskMetrics: a commitment to the education of all those who apply the model through clear assumptions and transparency of methods. Only by understanding the foundation of a model, and by knowing which assumptions are driven by practical needs and which by modeling exactitude, can the user know the realm of situations in which the model can be expected to perform well. This philosophy has motivated our restatement and clarification of the RiskMetrics modeling assumptions. Additionally, it has motivated us to discuss complementary modeling frameworks that may uncover sources of risk not revealed by the standard model. Chapters 3 (historical simulation) and 4 (stress testing) are thus included not as an obligatory nod to alternate approaches, but rather as necessary complements to the standard statistical model. Only through a combination of these is a complete picture of risk possible.

Throughout this document, our goal is the communication of our fundamental risk-modeling framework. However, in the interest of brevity, and to avoid overly taxing the patience of our readers, we have stayed away from delving into details that do not add to the basic understanding of our approach. For instance, in Chapter 5, we have chosen not to catalogue all of the instruments that we cover in our software application, but rather have provided a detailed look at a representative set of instruments that illustrate a broad range of pricing approaches: fixed cash flows, floating rate cash flows, options with closed-form pricing solutions, and options requiring Monte Carlo or tree-based pricing methods.

We recognize that following on *RiskMetrics Classic*, even if only in a focused treatment as we have written here, is a humbling task. We hope that this document is as useful in the year 2001 as *RiskMetrics Classic* was in 1996. To the readers of the old document, welcome back. We appreciate your continued interest, and thank you in particular for the feedback and questions over the last five years that have helped mold this new document. To those of you who are new to RiskMetrics in particular and risk modeling in general, we hope that this document gives you a solid understanding of the field, and happily invite questions, comments, and criticisms.

Part I

METHODOLOGIES FOR RISK MEASUREMENT

Chapter 1

Risk Factors

Risk management systems are based on models that describe potential changes in the factors affecting portfolio value. These "risk factors" are the building blocks for all pricing functions. In general, the factors driving the prices of financial securities are equity prices, foreign exchange rates, commodity prices, and interest rates. By generating future scenarios for each risk factor, we can infer changes in portfolio value and reprice the portfolio accordingly for different "states of the world."

One way to generate scenarios is to specify the probability that a risk factor will take a certain future value. We would then be making an assumption regarding its distribution. Another way is to look at the past behavior of risk factors and then assume that future behavior will be similar. These two alternatives lead to two different models to measure risk. The first is covered in Chapter 2, and is based on explicit assumptions about the probability distribution of risk factors. The second, based on the historical behavior of risk factors, will be explained in Chapter 3.

Once we have specified the possible scenarios and their likelihood, we produce profit and loss (P&L) scenarios. Part II covers the most important aspects of instrument pricing. Finally, we use the P&L scenarios to compute measures of risk for the portfolio. For example, we could calculate 95% Value at Risk as the loss amount that would be exceeded only 5% of the time. Part III describes the calculation of risk statistics and the creation of different kinds of reports based on those statistics.

In the rest of this chapter, we will describe the main types of risk factors, as well as the conventions we will adopt throughout the document to express them.

Equities

Equity risk factors are always expressed in the form of prices or levels. This means that an equity exposure can be represented by its own time series of prices, or alternatively, mapped to an appropriate index. For example, changes in value for a position in Ford stock can be described by changes in the stock itself or by its sensitivity to changes in an index such as the S&P 500.[1] Equity forwards and equity options are also driven by the same underlying price series.

In addition to straight equities, there are a range of instruments for which we have a history of prices, but not a good pricing function in terms of underlying risk factors. These instruments could be treated like equities instead of trying to work out a relationship between their price and the prices of a set of risk factors. An analogy in the straight equity world is to model an equity using only its own time series as opposed to the use of a factor model for equity pricing. Outside straight equities, we could model Brady bonds using their own time series of prices instead of utilizing a model based on an underlying yield curve.

Foreign exchange

Foreign exchange spot rates drive the currency risk of cash positions in foreign currencies, FX forwards, cross currency swaps, and FX options. Note that we can think of FX spot rates as the prices of unit amounts of foreign currencies. In most applications, we make use of covered parity between FX rates and interest rates to obtain forward currency prices.

Commodities

Commodity exposures are driven by spot and futures prices. Spot prices are used only for spot commodity transactions, while futures prices drive the commodity futures, commodity options, and options on commodity futures.

In the case of futures prices, we need to construct a constant maturity curve from the prices of individual contracts with specific delivery dates. The methodology used to construct constant maturity curves is described in Malz (2001a).

Interest Rates

The drivers of fixed income exposures can be expressed as zero-coupon bonds.[2] A zero-coupon bond is a simple fixed income security that pays one unit of local currency at maturity. The prices of zero-coupon bonds are directly linked to interest rates, and once we agree on a quoting convention for interest rates, we can obtain zero-coupon bond prices from interest rates. For example, zero-coupon interest rates in the U.S. are

[1] The sensitivity to an index is usually called the beta of the security with respect to the index.

[2] We can also think of zero-coupon bonds as discount factors.

usually quoted using semiannual compounding. If we denote the t-year semiannually compounded interest rate by $z_t^{(2)}$, then we can calculate the price of a zero-coupon bond maturing in t years as

$$B_t = \left(1 + \frac{z_t^{(2)}}{2}\right)^{-2t}. \tag{1.1}$$

If we now denote by z_t the t-year continuously compounded interest rate, we can express the price of the zero-coupon bond as

$$B_t = e^{-z_t t}. \tag{1.2}$$

Note that we can obtain continuously compounded rates from semiannual interest rates using the formula $z_t = 2\log(1 + z_t^{(2)}/2)$.

We use continuously compounded interest rates because they have nice properties that facilitate their mathematical treatment. For example, the logarithmic return on a zero-coupon bond is equal to the difference of interest rates multiplied by the maturity of the bond. That is,

$$\log\left(\frac{e^{-\widetilde{z}t}}{e^{-zt}}\right) = -(\widetilde{z} - z)t, \tag{1.3}$$

where $\widetilde{z}$ corresponds to a future interest rate scenario. This property of continuous rates turns out to be important because we can directly infer the behavior of changes in interest rates from the behavior of bond returns.

Next, using a set of zero-coupon bonds, we can price other fixed income securities such as coupon bonds and interest rate swaps. For example, suppose that we have a bond maturing in one year paying a semiannual coupon of 10%. Then, we can express the price of the bond as

$$P = 5B_{0.5} + 105B_1, \tag{1.4}$$

where B_t denotes the price of a zero-coupon bond maturing at time t. This means that the price P of the coupon bearing bond is a linear function of the prices of two zero-coupon bonds. The reader has probably noticed that this procedure is a little different from common market practice, where the price of the bond is expressed as a function of the yield for that particular security. We can write the price P of the bond in terms of the yield as

$$P = \frac{5}{(1 + \frac{y}{2})} + \frac{105}{(1 + \frac{y}{2})^2}. \tag{1.5}$$

Note that the yield y of the bond can also be obtained from the zero-coupon bond prices by combining (1.4) and (1.5). However, given the yield on a specific bond we cannot back out the zero-coupon rates.[3]

[3]This is because the yield represents an average return for an investment and hence incorporates security specific information such as coupon size, frequency, and time to maturity.

We can also apply the same process to price plain vanilla swaps. For example, if we assume for simplicity that we are entering into a swap today, then the floating leg will be worth par, and the fixed leg can be priced as a fixed coupon bond where the coupon is equal to the swap rate. Since the swap rate is chosen to make the value of the swap zero, this means that we can recover the swap rate from the term structure of zero-coupon rates.

We can use a set of bonds and swaps with maturities spanning the length of the term structure of interest rates to construct a yield curve. Malz (2001a) provides a detailed explanation of how to construct a yield curve from a set of bond and swap prices.[4] The advantage of constructing a yield curve is that we can recover the prices of a group of fixed income securities with a small set of information. That is, assuming that any intermediate point can be recovered by interpolation, a yield curve can be described with only a few nodes.This means that we can use a pre-specified set of zero-coupon bonds to price any fixed income instrument based on a reference yield curve. For example, if we had to price 50 different U.S. government bonds, we would price them using a common yield curve obtained from a fixed set of zero-coupon bonds.

The use of a yield curve does not guarantee that the price of every individual security will exactly match its market price, and adding a spread specific to each security might be necessary to match prevailing market prices. For example, if we know the current price of a bond, we solve for the spread such that when added to the base yield curve, we recover the price of the bond. This procedure is called calibration and is explained in detail in Chapter 5.

Although the yield curve gives us a good idea of the rate of borrowing for each term length, the liquidity and demand for a specific issue, as well as the credit quality of the issuer, introduce differences between the theoretical price and the observed market price of the security.[5] One way of minimizing this problem is to construct yield curves using only instruments issued by entities with similar credit quality. However, the construction of such corporate yield curves can be complicated due to the lack of bonds spanning the term length of the yield curve. For example, we can gather a set of BBB corporate bonds and use them to construct a representative yield curve for BBB corporate debt. Note that even when using corporate curves, we will still have pricing errors due to specific supply and demand effects. We can use a calibration procedure and obtain a specific spread on top of the corporate curve to try to minimize those pricing errors.

In addition to bonds and swaps, we can also price interest rate derivatives using yield curves expressed as a set of zero-coupon bond prices. For example, a bond option can be priced using Black's formula where the current value of the underlying can be obtained with the corresponding yield curve, and the only other relevant factor is the implied volatility of the bond. If we do not have a history of implied volatilities, we can assume that the implied volatility will remain constant throughout the analysis period. We can use similar approaches to price caps, floors, swaptions, and other interest rate derivatives. Chapter 5 presents a discussion of the models used to price these instruments.

[4] These methods also generally use money market rates to obtain the short end of the yield curve.

[5] The difference between the market price of a security and the price we obtain from the zero-coupon yield curve (i.e., the theoretical price) is the source of the specific risk.

While these four—equity prices, FX rates, commodity prices, and interest rates—are the main factors, there are others that influence prices such as implied volatility and credit spreads. In fact, every changing parameter in a pricing formula can be considered a risk factor. However, it is not always simple to specify the distribution of future values for every parameter. For example, in order to calculate the risk introduced in an option by changes in implied volatility, we need the corresponding time series of implied volatilities. In some cases, this information is not readily available.[6] Furthermore, two alternative sets of risk factors can describe equally well the changes in price for an instrument. For example, changes in a corporate bond can be explained by changes in a sovereign curve plus changes in the corresponding corporate spread. Alternatively, a change in the price of a corporate bond can also be attributed to changes in a corporate curve that embeds the credit quality information of the corporate issuer. In short, there are different ways to measure risk. Our framework will be extended to incorporate other risk factors and financial products.

The following two chapters introduce our model to describe the potential changes in risk factors, and the methods used to quantify the risks for a given portfolio.

[6] See Malz (2001a) for a complete discussion on implied volatilities.

Chapter 2

Models based on Distributional Assumptions

Chapter 2 discusses the distributional assumptions of the model and their role in the estimation of risk. The Monte Carlo and parametric methods are presented in detail, as well as the practical implications of their use, their differences, and their similarities.

2.1 The multivariate normal model for returns

One of the main contributions of the classic methodology presented in *RiskMetrics Classic* (see Morgan Guaranty Trust Company (1996)) is a model to update the return volatility estimates based on the arrival of new information, where the importance of old observations diminishes exponentially with time. Once we obtain a volatility estimate, the RiskMetrics methodology assumes that logarithmic returns on the risk factors follow a normal distribution conditional on the current volatility estimate.

The present chapter describes the distributional assumptions made in *RiskMetrics Classic*, as well as the RiskMetrics volatility model. We also show how to use these distributional assumptions in Monte Carlo methods to generate risk factor scenarios and compute risk statistics. Finally, we explain the parametric method originally described in *RiskMetrics Classic*.

As we mentioned above, the model for the distribution of future returns is based on the notion that logarithmic returns, when standardized by an appropriate measure of volatility, are independent across time and normally distributed.

Let us start by defining the logarithmic return of a risk factor as

$$r_{t,T} = \log\left(\frac{P_T}{P_t}\right) = p_T - p_t, \tag{2.1}$$

where $r_{t,T}$ denotes the logarithmic return from time t to time T, P_t is the level of the risk factor at time t, and $p_t = \log(P_t)$.[1]

Given a volatility estimate σ, the process generating the returns follows a geometric random walk:

$$\frac{\mathrm{d}P_t}{P_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t. \tag{2.2}$$

This means that the return from time t to time T can be written as

$$r_{t,T} = (\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma\varepsilon\sqrt{T-t}, \tag{2.3}$$

where $\varepsilon \sim N(0,1)$.[2]

There are two parameters that need to be estimated in (2.3): the drift μ and the volatility σ. In Kim, Malz and Mina (1999) we have shown that mean forecasts for horizons shorter than three months are not likely to produce accurate predictions of future returns. In fact, most forecasts are not even likely to predict the sign of returns for a horizon shorter than three months. In addition, since volatility is much larger than the expected return at short horizons, the forecasts of future distribution of returns are dominated by the volatility estimate σ. In other words, when we are dealing with short horizons, using a zero expected return assumption is as good as any mean estimate one could provide, except that we do not have to worry about producing a number for μ. Hence, from this point forward, we will make the explicit assumption that the expected return is zero, or equivalently that $\mu = \frac{1}{2}\sigma^2$.

We can incorporate the zero mean assumption in (2.3) and express the return as

$$r_{t,T} = \sigma\varepsilon\sqrt{T-t}, \tag{2.4}$$

The next question is how to estimate the volatility σ. We use an exponentially weighted moving average (EWMA) of squared returns as an estimate of the volatility. If we have a history of $m+1$ one-day returns from time $t-m$ to time t, we can write the one-day volatility estimate at time t as

$$\sigma = \frac{1-\lambda}{1-\lambda^{m+1}} \sum_{i=0}^{m} \lambda^i r_{t-i}^2 = R^\top R, \tag{2.5}$$

[1] The notation $\log(\cdot)$ is used to denote the natural logarithm.
[2] For a derivation of (2.3) see Hull (1997).

where $0 < \lambda \leq 1$ is the decay factor, r_t denotes the return from day t to day $t+1$, and

$$R = \sqrt{\frac{1-\lambda}{1-\lambda^{m+1}}} \begin{pmatrix} r_t \\ \sqrt{\lambda} r_{t-1} \\ \vdots \\ \sqrt{\lambda^m} r_{t-m} \end{pmatrix}. \tag{2.6}$$

The smaller the decay factor, the greater the weight given to recent events. If the decay factor is equal to one, the model reduces to an equally weighted average of squared returns. The obvious question to ask is: how large should the decay factor be?

By using the idea that the magnitude of future returns corresponds to the level of volatility, one approach to select an appropriate decay factor is to compare the volatility obtained with a certain λ to the magnitude of future returns.

In Section 5.3 of *RiskMetrics Classic*, we formalize this idea and obtain an optimal decay factor by minimizing the mean squared differences between the variance estimate and the actual squared return on each day. Using this method, we show that each time series (corresponding to different countries and asset classes), has a different optimal decay factor ranging from 0.9 to 1. In addition, we find that the optimal λ to estimate longer-term volatility is usually larger than the optimal λ used to forecast one-day volatility. The conclusion of the discussion in *RiskMetrics Classic* is that on average $\lambda = 0.94$ produces a very good forecast of one-day volatility, and $\lambda = 0.97$ results in good estimates for one-month volatility.[3]

An important consequence of using an exponential weighting scheme is that regardless of the actual number of historical returns used in the volatility calculation, the effective number of days used is limited by the size of the decay factor. In other words, 99.9% of the information is contained in the last $\log(0.001)/\log(\lambda)$ days. For example, if we use $\lambda = 0.94$, then 99.9% of the information is contained in the last 112 days. For $\lambda = 0.97$, 99.9% of the information is contained in the last 227 days.

The EWMA one-day volatility estimate changes every day—as we incorporate new information and discard old observations—reflecting the stochastic nature of volatility. To understand how volatility is changing through time in our model, we can write (2.5) in recursive form:

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1-\lambda) r_t^2, \tag{2.7}$$

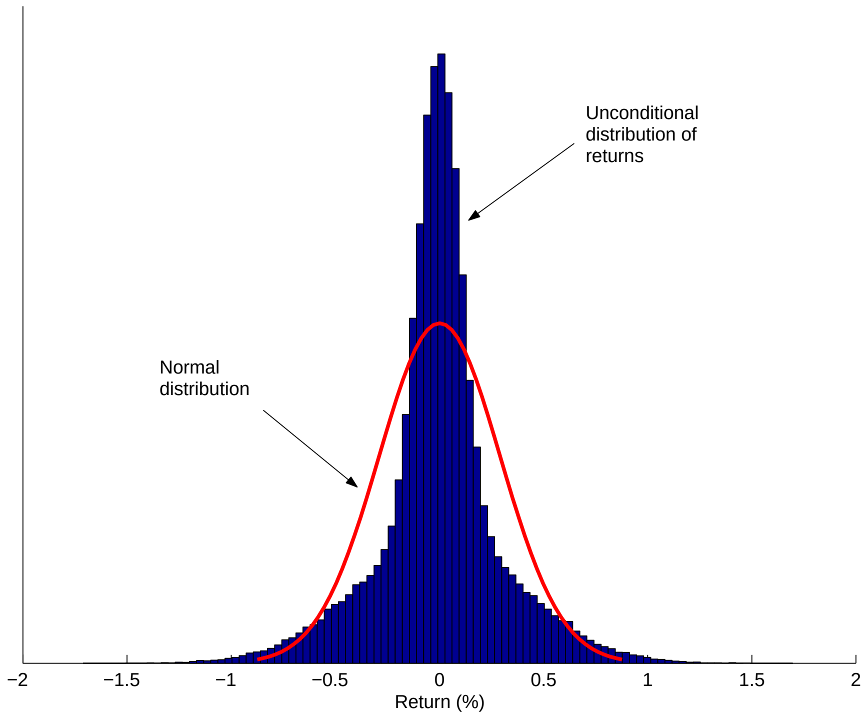
where σ_t denotes the volatility at time t.

From (2.7) we can see that the variance at time t is a weighted average of the variance at time $t-1$ and the magnitude of the return at time t. Note that the weight applied to the past variance is equal to λ, which is consistent with the notion that the decay factor is equivalent to the weight given to past observations.

Since volatility is not constant, it becomes very important to understand the difference between the conditional and the unconditional distribution of returns. The assumption behind our model is that one-day returns

[3] Fleming, Kirby and Ostdiek (2001) find an optimal decay factor close to 0.94 for one-day volatility using a similar approach.

Figure 2.1: **Conditional versus unconditional normal distributions**



conditioned on the current level of volatility are independent across time and normally distributed. It is important to note that this assumption does not preclude a heavy-tailed unconditional distribution of returns. For example, if returns on Mondays, Wednesdays and Fridays were independent and normally distributed with a volatility of 40%, and returns on Tuesdays and Thursdays were also independent and normally distributed, but with a volatility of 10%, we can only say that returns are normally distributed given that we know which day of the week it is. Unconditionally, the distribution of returns is non-normal and presents fat tails. Figure 2.1 compares the unconditional distribution of returns described above to a normal distribution with the same unconditional volatility. One can see that the unconditional distribution of returns has much heavier tails than those of a normal distribution.

The model explained above applies to a single risk factor. As we will illustrate, the model can be generalized to describe the dynamics of multiple risk factors.

Suppose that we have n risk factors. Then, the process generating the returns for each risk factor can be written as

$$\frac{dP_t^{(i)}}{P_t^{(i)}} = \mu_i dt + \sigma_i dW_t^{(i)}, \qquad i = 1, \ldots, n \tag{2.8}$$

where $\mathrm{Var}(dW^{(i)}) = dt$, $\mathrm{Cov}[dW^{(i)}, dW^{(j)}] = \rho_{i,j}dt$, and $\rho_{i,j}$ is the correlation between returns on assets i and j.

From (2.8) it follows that the return on each asset from time t to time $t+T$ can be written as

$$r_{t,T}^{(i)} = (\mu_i - \frac{1}{2}\sigma_i^2)(T-t) + \sigma_i\varepsilon_i\sqrt{T-t}, \tag{2.9}$$

where $\varepsilon_i \sim N(0,1)$, and $\mathrm{Cov}[\varepsilon_i, \varepsilon_j] = \rho_{i,j}$.

If we incorporate the zero mean assumption we get that

$$r_{t,T}^{(i)} = \sigma_i\varepsilon_i\sqrt{T-t}. \tag{2.10}$$

We can see that the equations representing the evolution of returns over time are almost identical for a single or multiple risk factors ((2.4) and (2.10) respectively). The only difference is that when we have more than one risk factor, we need to take into account the correlation between returns on the various risk factors.

As in the single risk factor case, we need to obtain an estimate of the future variability of returns. We also need to assess how close each pair of risk factors move together by estimating their correlation. This information is summarized in a covariance matrix that we denote by Σ. Each entry on the covariance matrix represents the covariance between each pair of assets and is equal to the product of their respective volatilities and their correlation. For example, the covariance between returns on asset i and asset j can be written as:

$$\Sigma_{i,j} = \sigma_i\sigma_j\rho_{i,j} = \frac{1-\lambda}{1-\lambda^{m+1}}\sum_{k=0}^{m}\lambda^k r_{t-k}^{(i)} r_{t-k}^{(j)}. \tag{2.11}$$

Note that (2.11) is similar to (2.5), where we have replaced the squared return on a single asset with the product of the returns on two assets. This means that if the return for both assets is either positive or negative on the same days, the correlation between returns will be positive. If the returns tend to be of opposite signs on the same days, then the correlation will be negative.

We can also write the covariance matrix Σ as

$$\Sigma = R^\top R, \tag{2.12}$$

where R is an $m \times n$ matrix of weighted returns:

$$R = \sqrt{\frac{1-\lambda}{1-\lambda^{m+1}}}\begin{pmatrix} r_t^{(1)} & r_t^{(2)} & \cdots & r_t^{(n)} \\ \sqrt{\lambda}r_{t-1}^{(1)} & \cdots & \cdots & \sqrt{\lambda}r_{t-1}^{(n)} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \sqrt{\lambda^m}r_{t-m}^{(1)} & \sqrt{\lambda^m}r_{t-m}^{(2)} & \cdots & \sqrt{\lambda^m}r_{t-m}^{(n)} \end{pmatrix}. \tag{2.13}$$

In this section we have stated the distributional assumptions for risk factor returns and explained how to calculate the volatility and correlation parameters from historical data. In the following section we will show how to use simulation methods to obtain risk factor return scenarios from these distributions. We will also describe how to use risk factor scenarios to arrive at P&L scenarios for an entire portfolio.

2.2 Monte Carlo simulation

In order to understand the process to generate random scenarios, it is helpful to write (2.8) in terms of independent Brownian increments $d\widetilde{W}^{(i)}$:

$$\frac{dP_t^{(i)}}{P_t^{(i)}} = \mu_i dt + \sum_{j=1}^{n} c_{ji} d\widetilde{W}_t^{(j)}. \qquad i = 1, \ldots, n \tag{2.14}$$

The process of going from (2.8) to (2.14) is similar to a principal component analysis, where we can write a set of correlated variables as the linear combination of a set of independent variables.[4] The coefficients of the linear combination c_{ji} are not unique but satisfy certain requirements.

We can gain more intuition about the coefficients c_{ji} if we write (2.14) in vector form:

$$\frac{\mathbf{dP}_t}{\mathbf{P}_t} = \mu dt + C^\top \mathbf{d\widetilde{W}}_t, \tag{2.15}$$

where $\{\frac{\mathbf{dP}_t}{\mathbf{P}_t}\}_i = \frac{dP_t^{(i)}}{P_t^{(i)}}$ $(i = 1, 2, \ldots, n)$ is a $n \times 1$ vector, $\mathbf{d\widetilde{W}}$ is a vector of n independent Brownian increments, and $C = [c_{ij}]$ is any $n \times n$ matrix such that the covariance matrix of returns Σ can be written as $\Sigma = C^\top C$.[5]

This means that the vector of returns for every risk factor from time t to time T can be written as

$$\mathbf{r}_{t,T} = (\mu - \frac{1}{2}\sigma^2)(T - t) + C^\top \mathbf{z}\sqrt{T - t}, \tag{2.16}$$

where $\mathbf{r}_{t,T}$ is a vector of returns from from time t to time T, σ^2 is a $n \times 1$ vector equal to the diagonal of the covariance matrix Σ, and $\mathbf{z} \sim MVN(0, \mathbf{I})$.[6]

Following our assumption that $\mu_i = \frac{1}{2}\sigma_i^2$, we can rewrite (2.16) as

$$\mathbf{r}_{t,T} = C^\top \mathbf{z}\sqrt{T - t}, \tag{2.17}$$

[4] In other words, this is just a rotation resulting in orthogonal components.

[5] The importance of C will become apparent when we explain the generation of multivariate returns.

[6] MVN stands for multivariate normal.

Note that the one-day returns ($T - t = 1$) from (2.17) follow a MVN distribution with zero mean and covariance matrix Σ. This means that (2.10) and (2.17) are equivalent. To verify this statement, we can calculate the covariance of $\mathbf{r}$ as

$$\begin{aligned} \text{Covariance} &= \mathbf{E}[C^\top \mathbf{z}\mathbf{z}^\top C] && (2.18)\\ &= C^\top \mathbf{E}[\mathbf{z}\mathbf{z}^\top] C && (2.19)\\ &= C^\top \mathbf{I} C && (2.20)\\ &= \Sigma. && (2.21) \end{aligned}$$

Following (2.4) and (2.17) we can use independent standard normal variables to generate return scenarios. For instance, if we only have one risk factor, we can follow (2.4) and generate a T-day return by multiplying a standard normal variable ε by the scaled volatility $\sqrt{T}\sigma$.

In the case where we want to generate scenarios of joint returns for multiple risk factors, we first need to find a matrix C such that $\Sigma = C^\top C$. Then, we generate n independent standard normal variables that we store in a column vector $\mathbf{z}$. Finally, we multiply the scaled volatility matrix $\sqrt{T}C^\top$ by the vector $\mathbf{z}$ to produce a $n \times 1$ vector $\mathbf{r}$ of T-day joint returns.

It is important to emphasize that the choice of C is not unique. There are different methods to decompose Σ that result in different C. Two popular methods for decomposing a matrix are the Cholesky decomposition and the Singular Value decomposition (SVD).[7] One important difference between these decompositions is that the Cholesky algorithm fails to provide a decomposition when the covariance matrix is not positive definite. On the other hand, we can always find the SVD of a matrix. A non-positive definite covariance matrix corresponds to a situation where at least one of the risk factors is redundant, meaning that we can reproduce the redundant risk factor as a linear combination of the other risk factors. This situation most typically arises when the number of days used to calculate the covariance matrix is smaller than the number of risk factors. For example, if we have three risk factors, but only two possible states of the world, then we can hedge one of the risk factors with a combination of the other two.

Once we have produced return scenarios for the risk factors, we need to translate those returns into profit and loss scenarios for the instruments that we hold. For example, if we hold an equity, and have generated a one-day return scenario r, we can express the one-day profit and loss (P&L) of the position as $P_1 - P_0$, where P_0 is the current equity price, and $P_1 = P_0 e^r$ is the price of the equity one day from now. In a similar fashion, if we are holding an equity option, we can obtain the P&L as $BS(P_1) - BS(P_0)$, where $BS(P)$ is the Black-Scholes pricing formula evaluated at the equity price P.

In general, if we have M instruments in a portfolio, where the present value of each instrument is a function of n risk factors $V_j(\mathbf{P})$, with $j = 1, \ldots, M$ and $\mathbf{P} = (P^{(1)}, P^{(2)}, \ldots, P^{(n)})$, we can obtain a one-day P&L scenario for the portfolio following the next steps:

1. Generate a set $\mathbf{z}$ of independent standard normal variables.

[7] Trefethen, L.N. and Bau, D. III (1997) is a good reference for the calculation of Cholesky and SVD.

2. Transform the independent standard normal variables into a set of returns $\mathbf{r} = r^{(1)}, r^{(2)}, \ldots, r^{(n)}$ corresponding to each risk factor using the matrix C. In other words, $\mathbf{r} = C^\top \mathbf{z}$.

3. Obtain the price of each risk factor one day from now using the formula $P_1 = P_0 e^r$.

4. Price each instrument using the current prices $\mathbf{P}_0$ and the one-day price scenarios $\mathbf{P}_1$.

5. Get the portfolio P&L as $\sum_j (V_j(\mathbf{P}_1) - V_j(\mathbf{P}_0))$.

The procedure can be easily extended to generate T-day P&L scenarios. The only difference is that we generate a T-day price scenario using the formula $P_T = P_0 e^{r\sqrt{T}}$.

As we just discussed, the simulation of returns scenarios is based on the generation of independent and identically distributed standard normal random variables. The first step to generate normally distributed random variables is to generate uniform random numbers taking values from zero to one with equal probability. Once we have independent uniform random numbers, we apply a transformation to obtain normally distributed random numbers. Algorithms for computer random number generation have been widely studied, and their two most desirable properties are a long period (a sequence of random numbers should not repeat too often) and negligible serial correlations (e.g., small numbers should not always be followed by small numbers). The random number generator that we have used in our applications is a version of an algorithm proposed by L'Ecuyer (1988) with a period of 2×10^{18}, and where we shuffle the observations to break up serial correlations.[8] In practice, if we obtain 10,000 scenarios on 100 risk factors every day, it would take about 8 billion years to repeat the same set of scenarios.[9]

This concludes our discussion on random scenario generation. Up to this point, we have stated the distributional assumptions for our model (returns are conditionally MVN), and explained how to estimate the relevant parameters (the covariance matrix Σ) from historical data. Then, we discussed how to sample from our assumed distribution of returns to obtain P&L scenarios. Note that as long as we can express the price of a given instrument as a function of the risk factors, we can use our framework to obtain P&L scenarios for that instrument (and consequently for the entire portfolio). In Chapter 6, we explain how to use the generated P&L scenarios to calculate various risk measures (including VaR, Marginal Var, and Incremental Var).

The risk analysis in this section was based on the generation of sample scenarios for each instrument in the portfolio. Note that the independence between our distributional assumptions and the specific portfolio pricing functions allows for a tremendous flexibility in Monte Carlo methods. The price for this flexibility is computational complexity. If we are willing to sacrifice some accuracy, and incorporate additional assumptions about the behaviour of the pricing functions, we can avoid some of the computational burden of Monte Carlo methods and come up with a simple parametric method for risk calculations. Parametric

[8] See Press, Teukolsky, Vetterling and Flannery (1992) pp. 278-283 for a technical description of the algorithm.

[9] One can also make use of variance reduction techniques to reduce the number of simulations required to achieve a certain precision level. Three of these techniques are importance sampling, stratified sampling, and control variates. For a reference on variance reduction techniques in a VaR framework see Glasserman, Heidelberger and Shahabuddin (2000b).

methods were originally described in the classic RiskMetrics methodology. The following section provides a detailed explanation of the parametric methods.

2.3 Parametric methods

As we mentioned in the previous section, the parametric approach represents an alternative to Monte Carlo simulation to calculate risk measures. Parametric methods present us with a tradeoff between accuracy and speed. They are much faster than Monte Carlo methods, but not as accurate unless the pricing function can be approximated well by a linear function of the risk factors.

The idea behind parametric methods is to approximate the pricing functions of every instrument in order to obtain an analytic formula for VaR and other risk statistics. In this section, we describe the so called delta method which is based on a linear approximation of the pricing functions.

Let us assume that we are holding a single position dependent on n risk factors denoted by $P^{(1)}, P^{(2)}, \ldots, P^{(n)}$. To calculate VaR, we approximate the present value V of the position using a first order Taylor series expansion:

$$V(\mathbf{P} + \Delta\mathbf{P}) \approx V(\mathbf{P}) + \sum_{i=1}^{n} \frac{\partial V}{\partial P^{(i)}} \Delta P^{(i)}. \tag{2.22}$$

Equation (2.22) can be interpreted in the following way. If the price of one of the risk factors changes by an amount equal to ΔP, then the present value of the position will approximately change by the sensitivity of the position to changes in that risk factor $(\partial V/\partial P)$ weighted by the magnitude of the change (ΔP). To take into account simultaneous shocks to different risk factors, we add all the individual increments $(\partial V/\partial P^{(i)})\Delta P^{(i)}$.

From (2.22) we can then approximate the change in present value as

$$\Delta V = V(\mathbf{P} + \Delta\mathbf{P}) - V(\mathbf{P}) \approx \sum_{i=1}^{n} \delta_i r^{(i)}, \tag{2.23}$$

where

$$\delta_i = P^{(i)} \frac{\partial V}{\partial P^{(i)}}. \tag{2.24}$$

Note that (2.23) gives a simple expression for the P&L as a linear combination of risk factor returns. It is convenient to express (2.23) in matrix notation:

$$\Delta V \approx \delta^{\top}\mathbf{r}. \tag{2.25}$$

The entries of the δ vector are called the "delta equivalents" for the position, and they can be interpreted as the set of sensitivities of the present value of the position with respect to changes in each of the risk factors.

Note that the returns in (2.23) are actually percentage returns ($r = \Delta P/P$), but our model is constructed on the assumption that logarithmic returns are normally distributed. To be consistent with our distributional assumptions, we make the assumption that $\log(P_1/P_0) \approx P_1/P_0 - 1$. [10]

In order to understand our choice to model logarithmic returns, we need to explore the properties of percentage and logarithmic returns. Percentage returns have nice properties when we want to aggregate across assets. For example, if we have a portfolio consisting of a stock and a bond, we can calculate the return on the portfolio as a weighted average of the returns on each asset:

$$\frac{P_1 - P_0}{P_0} = wr^{(1)} + (1-w)r^{(2)}, \tag{2.26}$$

where w is the proportion of the portfolio invested in the stock, $r^{(1)} = (S_1 - S_0)/S_0$ is the return on the stock, and $r^{(2)}$ is the return on the bond. On the other hand, it is easy to verify that the logarithmic return on a portfolio is not a weighted average of the individual asset logarithmic returns. This property implies that the rigorous way of writing the change in value of the portfolio in (2.23) is as a function of percentage returns. However, we have chosen to express (2.23) in terms of logarithmic returns. The reasons for our choice are explained below.

In contrast with percentage returns, logarithmic returns aggregate nicely across time. The logarithmic return from time t to time T is equivalent to the sum of the logarithmic return from time t to time τ and the logarithmic return from time τ to time T, where $t \leq \tau \leq T$. This can be shown easily using the standard properties of logarithms:

$$r_{t,T} = \log\left(\frac{P_T}{P_t}\right) = \left(\log\frac{P_\tau}{P_t}\right) + \left(\log\frac{P_T}{P_\tau}\right) = r_{t,\tau} + r_{\tau,T}. \tag{2.27}$$

This temporal additivity of logarithmic returns implies that if one-period returns are independent (as implied by the geometric random walk of (2.2)) the volatility of returns scales with the square root of time. After considering the advantages of each type of return, it is clear that we have to forego either the ability to aggregate across assets or the ability to aggregate across time (implying that we cannot scale volatility as the square root of time). Looking at the parametric method in isolation, the choice is not obvious. However, if we consider that Monte Carlo simulation does not benefit from linear aggregation across returns, that volatility scaling is an important property of the model for both Monte Carlo and parametric methods, and that the consistency of distributional assumptions across methods is desirable, we conclude that logarithmic returns offer the best alternative.

Delta equivalents have nice aggregation properties. Suppose that we have a portfolio consisting of M

[10] This approximation is particularly good when returns are small ($P_1/P_0 \approx 1$), because $\log(1+x) \approx x$ when x is small.

positions. Then, the P&L of the total portfolio can be written as

$$\text{P\&L} = \sum_{j}^{M} \Delta V_j \tag{2.28}$$

$$\approx \sum_{j}^{M} \delta_j^\top \mathbf{r} \tag{2.29}$$

$$= \delta_{\text{Portfolio}}^\top \mathbf{r}. \tag{2.30}$$

This means that we can calculate the delta equivalents independently for each position and then aggregate them to obtain the delta equivalents for the total portfolio.

Example 2.1 Delta equivalents of an equity denominated in a foreign currency

If we are a USD based investor, the delta equivalents for a stock denominated in GBP with a current price of GBP 7, and an exchange rate of GBP 0.7 per USD, are USD 10 for both the foreign exchange and the equity risk factors. These values can be formally derived from (2.24) and the formula for the present value of the investment in USD. Denoting the value of the position by V, the foreign exchange rate by S^{FX}, and the equity price by S^{EQ}, we have that $V = S^{FX} \times S^{EQ} =$ USD 10, $\partial V/\partial S^{FX} = S^{EQ}$, $\partial V/\partial S^{EQ} = S^{FX}$, and hence $V = \delta_{S^{FX}} = \delta_{S^{EQ}} =$ USD 10. ■

Example 2.2 Delta equivalents of a bond

Let us say that we are a EUR based investor and have EUR 100 invested in a two-year zero-coupon bond.[11] Since the two-year zero-coupon bond is a risk factor itself, we have that $\delta =$ EUR $100B_2$, where B_2 is the present value of a zero-coupon bond expiring in two years. Similarly, if we have EUR 100 invested on a two-year bond paying an annual coupon of 6%, we have that the delta equivalent with respect to the one-year zero-coupon bond is EUR $6B_1$, and the delta equivalent with respect to the two-year zero-coupon bond is EUR $106B_2$.

If we had the one-year zero-coupon bond and the two-year coupon bond in a portfolio, we could use the additivity property to calculate the delta equivalents for the portfolio. In this case, the delta equivalent of the portfolio with respect to the one-year zero-coupon bond would be EUR $6B_1$, and the delta equivalent with respect to the two-year zero-coupon bond would be EUR $206B_2$.

Note that the cash flows in our examples fall exactly on a risk factor (i.e., the one-year and the two-year zero-coupon bonds). If we had a cash flow at 1.5 years, we would have to map the present value of the cash flow between the two adjacent nodes at one and two years. Cash flow maps are explained in Chapter 5. ■

So far, we have been able to express the P&L of a portfolio as a linear combination of risk factor returns, and it seems only fair to ask if we have gained anything through this exercise. Since our final objective in

[11] Note that in this example we do not have any foreign exchange risk.

this section is to obtain statistics without relying on simulation methods, we need to obtain a tractable P&L distribution.

Since risk factor returns are normally distributed, it turns out that the P&L distribution under our parametric assumptions is also normally distributed with mean zero and variance $\delta^\top \Sigma \delta$.[12] In other words,

$$\Delta V \sim N(0, \delta^\top \Sigma \delta). \tag{2.31}$$

The normal distribution is entirely described by its mean and variance, so the fact that the P&L is normally distributed has deep implications for the calculation of risk measures. For example, percentiles of a normal distribution can be expressed as multiples of the standard deviation, and we can therefore conclude that VaR under our parametric assumptions will be a multiple of the standard deviation ($\sqrt{\delta^\top \Sigma \delta}$). In Chapter 6, we explain how to calculate various risk measures (including VaR, Marginal Var, and Incremental Var), using the assumptions set forth in this section.

In this chapter we provide a description of a risk model based on explicit distributional assumptions for factor returns. Section 2.1 states the multivariate normal model for the distribution of risk factor returns, while Section 2.2 and Section 2.3 explain two different methods to assess risk based on these distributions. The first method is based on Monte Carlo simulation and makes no assumption regarding the pricing functions for the instruments. Monte Carlo methods are highly accurate, but are also computationally expensive. The second method is based on a parametric analysis that relies on the assumption that pricing functions are linear in the risk factors. Parametric methods provide very fast answers that are only as accurate as the underlying linearity assumption.

An alternative to an explicit model for return distributions is the use of historical frequencies of returns. The main advantage of using the empirical distribution of risk factor returns is that no specific distributional assumptions need to be made and no parameters (e.g., volatilities and correlations) need to be estimated. This means that the historical data dictate the shape of the multivariate distribution of returns. The main shortcoming of using an empirical distribution is that the period selected might not be representative of potential future outcomes.

[12] The result follows from the fact that linear combinations of normal variables are also normally distributed.

Chapter 3

Models Based on Empirical Distributions

A shared feature of the methods of Chapter 2 is that they rely on the assumption of a conditionally normal distribution of returns. However, it has often been argued that the true distribution of returns (even after standardizing by the volatility) implies a larger probability of extreme returns than that implied from the normal distribution. Although we could try to specify a distribution that fits returns better, it would be a daunting task, especially if we consider that the new distribution would have to provide a good fit across all asset classes. There has been a lot of discussion of these issues and some alternative distributions have been proposed. But up until now academics as well as practitioners have not agreed on an industry standard heavy-tailed distribution (or family of distributions).[1]

Instead of trying to explicitly specify the distribution of returns, we can let historical data dictate the shape of the distribution. In other words, we can come up with an empirical distribution of risk factor returns from the frequency with which they are observed. This means that if returns larger than 10% have occurred on average on one out of 20 days in the past, we say that there is a 5% probability that tomorrow's return is larger than 10%. In this approach, the historically observed risk factor changes are assumed to be independent and identically distributed (i.i.d.), and correspond to the same distribution applicable to the forecast horizon.

It is important to emphasize that while we are not making direct assumptions about the likelihood of certain events, those likelihoods are determined by the historical period chosen to construct the empirical distribution of risk factors. Therefore, the choice of the length of the historical period is a critical input to the historical simulation model. In the selection of a sample period, we are faced with a trade-off between long sample periods which potentially violate the assumption of i.i.d. observations (due to regime changes) and short sample periods which reduce the statistical precision of the estimates (due to lack of data). The problem with using old information is that it might not be relevant in the current regime. One way of mitigating this problem is to scale past observations by an estimate of their volatility. Hull and White (1998) present a volatility updating scheme; instead of using the actual historical changes in risk factors, they use historical

[1] See Appendix A for a discussion of non-normal distributions.

changes that have been adjusted to reflect the ratio of the current volatility to the volatility at the time of the observation. As a general guideline, if our horizon is short (one day or one week), we should use the shortest possible history that provides enough information for reliable statistical estimation.[2] It is also important to note that regulators usually require the use of at least one year of data in VaR calculations.[3]

3.1 Historical simulation

One can make use of the empirical distribution of returns and obtain risk statistics through the use of historical simulation. The premise behind historical simulation is that potential changes in the underlying risk factors are identical to the observed changes in those factors over a defined historical period. This means that we perform a historical simulation by sampling from past returns, and applying them to the current level of the risk factors to obtain risk factor price scenarios. We finally use these price scenarios to obtain P&L scenarios for the portfolio. This historical simulation approach has the advantage of reflecting the historical multivariate distribution of risk factor returns. Note that this method also incorporates information about extreme returns as long as they are included in our sample period.

To formalize these ideas, suppose that we have n risk factors, and that we are using a database containing m daily returns. Let us also define the $m \times n$ matrix of historical returns as

$$R = \begin{pmatrix} r_t^{(1)} & r_t^{(2)} & \cdots & r_t^{(n)} \\ r_{t-1}^{(1)} & \cdots & \cdots & r_{t-1}^{(n)} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ r_{t-m}^{(1)} & r_{t-m}^{(2)} & \cdots & r_{t-m}^{(n)} \end{pmatrix}. \tag{3.1}$$

Then, as each return scenario corresponds to a day of historical returns, we can think of a specific scenario $\mathbf{r}$ as a row of R.

Now, if we have M instruments in a portfolio, where the present value of each instrument is a function of the n risk factors $V_j(\mathbf{P})$, with $j = 1, \ldots, M$ and $\mathbf{P} = (P^{(1)}, P^{(2)}, \ldots, P^{(n)})$, we can obtain a T-day P&L scenario for the portfolio as follows:

1. Take a row $\mathbf{r}$ from R corresponding to a return scenario for each risk factor.
2. Obtain the price of each risk factor T days from now using the formula $P_T = P_0 e^{r\sqrt{T}}$.
3. Price each instrument using the current prices $\mathbf{P}_0$ and also using the T-day price scenarios $\mathbf{P}_T$.

[2]For a discussion of the construction of precision measures in historical simulation, see Butler and Schachter (1998).
[3]See Basel Committee on Banking Supervision (1996).

4. Get the portfolio P&L as $\sum_j (V_j(\mathbf{P}_T) - V_j(\mathbf{P}_0))$.

This process is almost identical to the procedure outlined in Section 2.2 for Monte Carlo simulation, except that instead of sampling from a normal distribution, we sample the returns from our historical database.

Note that in order to obtain a T-day return scenario in step 2, we multiplied the one-day return scenario r by $\sqrt{T}$. This guarantees that the volatility of returns scales with the square root of time. In general, this scaling procedure will not exactly result in the true T-day return distribution, but is a practical rule of thumb consistent with the scaling in Monte Carlo simulation. An alternative method would be to create a set of T-day non-overlapping returns from the daily return data set. This procedure is theoretically correct, but it is only feasible for relatively short horizons because the use of non-overlapping returns requires a long data history. For example, if we have two years of data and want to estimate the distribution of one-month returns, the data set would be reduced to 24 observations, which are not sufficient to provide a reliable estimate. It is important to mention that in this case, the use of overlapping returns does not add valuable information for the analysis and introduces a bias in the estimates. The intuition is that by creating overlapping returns, a large observation persists for $T - 1$ windows, thus creating $T - 1$ large returns from what otherwise would be a single large return. However, the total number of observations increases by roughly the same amount, and hence the relative frequency of large returns stays more or less constant. In addition, the use of overlapping returns introduces artificial autocorrelation, since large T-day overlapping returns will tend to appear successively.

Example 3.1 Obtaining the portfolio P&L scenarios from historical returns

Suppose that we are a USD based investor and have a portfolio consisting of a cash position of EUR one million, 13,000 shares of IBM, and a short position consisting of a one year at-the-money call on 20,000 shares of IBM. The current exchange rate is USD 0.88 per EUR, the price of IBM is USD 120 per share, the one year rate is 6%, and the implied volatility is 45.62%. The current value of the portfolio is then USD 1,946,123. Table 3.1 shows the current value of each position.

Table 3.1: Current value of the portfolio

Position	Value (USD)
Cash	880,000
Equity	1,560,000
Option	-493,876
Total	1,946,123

We can apply historical return scenarios to our positions and obtain one-day portfolio P&L scenarios. Table 3.2 contains historical returns for three consecutive days.

Table 3.2: Historical returns

Date	EUR	IBM	1Y Bond
22-Sep-00	3.74%	1.65%	0.04%
21-Sep-00	0.56%	-1.35%	-0.05%
20-Sep-00	0.18%	0.60%	0.00%

For example, taking the historical returns for 22-Sep-2000, we can see that our position in EUR would have gained 3.74% and its P&L would have been USD 880,000 $\times$ $[e^{0.0374} - 1] =$ USD 33,535. Similarly, we can calculate the P&L for the equity position as USD 1,560,000 $\times$ $[e^{0.0165} - 1] =$ USD 25,953. We finally need to compute the P&L of the option position. To do this, we need to calculate the new price of the underlying equity (IBM) and the new interest rate based on the returns on 22-Sep-2000. The new price of IBM is USD 120 $\times$ $e^{0.0165} =$ USD 121.99, and the new interest rate is 6% $-$ 0.04% $=$ 5.96%. We can then use the Black-Scholes pricing formula with the new IBM price and discount rate to obtain the P&L of the option position as USD 20,000 $\times$ $[BS(120, 6\%) - BS(121.99, 5.96\%)] = -$USD 25,411. To calculate the P&L for the total portfolio, we simply sum the individual P&L's for each position.

We can repeat this exercise for each day to obtain a set of one-day portfolio P&L historical scenarios. Table 3.3 contains the P&L scenarios for the portfolio corresponding to each day of historical returns.

Table 3.3: P&L historical scenarios

Date	P&L (USD)
22-Sep-2000	34,078
21-Sep-2000	3,947
20-Sep-2000	1,688

■

In this chapter we explained how to obtain P&L scenarios for a portfolio from the risk factors' historical return scenarios. By using actual historical returns instead of simulated returns from a predetermined distribution, we can capture the fat tails often found in many risk factors' return distributions. As we will describe in Chapter 6, once we have a set of P&L scenarios, the computation of risk measures is independent of whether we used Monte Carlo or historical simulation to obtain those scenarios. In Chapter 6, we will give a detailed account of how to use the historical P&L scenarios to calculate risk measures.

In the following chapter we introduce stress testing as a complement to the statistical methods presented in Chapters 2 and 3. The advantage of stress tests is that they are not based on statistical assumptions about the distribution of risk factor returns. Since any statistical model has inherent flaws in its assumptions, stress tests are considered a good companion to any statistically based risk measure.

Chapter 4

Stress Testing

Stress tests are intended to explore a range of low probability events that lie outside of the predictive capacity of any statistical model. These events might be related to crises produced by major displacements such as wars, political instability, natural catastrophes, or speculative attacks on currencies; they could also be linked to changes in return expectations or the risk appetite of investors, possibly leading to the burst of a speculative bubble; or they could be attributable to events that are somewhat foreseeable such as shifts in monetary policy. Therefore, the estimation of the potential economic loss under hypothetical extreme changes in risk factors allows us to obtain a sense of our exposure in abnormal market conditions. For example, we can investigate how much we would lose in the event of an equity crisis of a magnitude comparable to the 1987 stock market crash. Alternatively, we could gauge the potential effect of a currency crisis on our portfolio, such as the Brazilian real devaluation of 1998.

Stress tests can be done in two steps.

1. Selection of stress events. This is the most important and challenging step in the stress testing process. The goal is to come up with credible scenarios that expose the potential weaknesses of a portfolio under particular market conditions.

2. Portfolio revaluation. This consists of marking-to-market the portfolio based on the stress scenarios for the risk factors, and is identical to the portfolio revaluation step carried out under Monte Carlo and historical simulation for each particular scenario. Once the portfolio has been revalued, we can calculate the P&L as the difference between the current present value and the present value calculated under the stress scenario.

The most important part of a stress test is the selection of scenarios. Unfortunately, there is not a standard or systematic approach to generate scenarios and the process is still regarded as more of an art than a science.[1]

[1] For a reference on generation of stress scenarios see Laubsch (1999). Malz (2000) constructs a warning signal for stress events based on implied volatility.

Given the importance and difficulty of choosing scenarios, we present three options that facilitate the process: historical scenarios, simple scenarios, and predictive scenarios.

4.1 Historical scenarios

A simple way to develop stress scenarios is to replicate past events. For instance, we could take the market changes experienced during the Russian crisis, and investigate the effect that they would have in our current portfolio. In other words, a historical stress test could answer the question: what would happen to my portfolio if the events that caused Russia to default happened again?

One can select a historical period spanning a financial crisis (e.g., Black Monday (1987), Tequila crisis (1995), Asian crisis (1997), Russian crisis (1998)) and use the returns of the risk factors over that period as the stress scenarios. In general, if the user selects the period from time t to time T, then following (2.1) we calculate the historical returns as

$$r = \log\left(\frac{P_T}{P_t}\right), \tag{4.1}$$

and calculate the P&L of the portfolio based on the calculated returns

$$\text{P\&L} = V(Pe^r) - V(P). \tag{4.2}$$

Example 4.1 The Russian crisis

Let us say that we have USD 3,000 invested in a portfolio equally distributed among the Brazilian, Indonesian, and Polish stock markets. We can then calculate the impact that an event of the magnitude of the Russian crisis would have on our portfolio. In this example, we take the historical returns on the relevant markets between 1-Jul-1998 and 30-Aug-1998 and apply them to our portfolio. Table 4.1 shows the logarithmic returns between those dates.

Table 4.1: Returns between 1-Jul-1998 and 30-Aug-1998

	Equity		Foreign Exchange	
Brazil	Bovespa	-48.19%	BRL	-1.34%
Indonesia	JSE	-36.47%	IDR	22.60%
Poland	WIG	-41.24%	PLN	-10.19%

To calculate the change in value for each of our positions, we use (4.2). If we denote by V the current value of our position in a foreign index, $r^{(1)}$ denotes the logarithmic return on the equity index (in local currency), and $r^{(2)}$ corresponds to the logarithmic return on the foreign exchange rate, we have that the change in value of each of our positions is given by:

$$\Delta V = V[e^{r^{(1)}+r^{(2)}} - 1], \tag{4.3}$$

where $V =$ USD 1,000 per instrument in our example.

Following (4.3), we can compute the change in value of our portfolio under the historical stress event corresponding to the Russian crisis. Table 4.2 presents the resulting changes in value by position.

Table 4.2: Historical stress test results

Country	Change in Value (USD)
Brazil	-390.59
Indonesia	-129.58
Poland	-402.11
Total	-922.29

■

4.2 User-defined simple scenarios

We have seen that historical extreme events present a convenient way of producing stress scenarios. However, historical events need to be complemented with user-defined scenarios in order to span the entire range of potential stress scenarios, and possibly incorporate expert views based on current macroeconomic and financial information.

In the simple user-defined stress tests, the user changes the value of some risk factors by specifying either a percentage or absolute change, or by setting the risk factor to a specific value. The risk factors which are unspecified remain unchanged. Then, the portfolio is revalued using the new risk factors (some of which will remain unchanged), and the P&L is calculated as the difference between the present values of the portfolio and the revalued portfolio.

Example 4.2 Simple stress test: a currency crisis

Using the emerging markets equity portfolio of Example 4.1, we can show how a simple stress test works given a user-defined scenario. The scenario in this case is a currency crisis where each currency devalues by 10%. Under this scenario, all the losses are due to changes in the foreign exchange rates and we keep the equity indices (in local terms) constant.

Since we are only moving the FX rates, the change in value of the portfolio is given by

$$\Delta V = V[e^{r^{(2)}} - 1], \tag{4.4}$$

where $r^{(2)} = \log(1 - 0.1) = -0.1053$ corresponds to a 10% depreciation expressed in logarithmic returns. The stress test results from the user-defined scenario are given in Table 4.3. The total P&L resulting from the scenario is simply the sum of the P&L's for the individual instruments.

Table 4.3: Simple stress test results

Country	Change in Value (USD)
Brazil	-100
Indonesia	-100
Poland	-100
Total	-300

Note that the results from this stress test do not reflect the effect that a currency crisis would have in the equity markets. The next section explains how to incorporate the effect that the user-stressed risk factors (also called core factors) have on the remaining risk factors (also called peripheral factors). ■

4.3 User-defined predictive scenarios

Since market variables tend to move together, we need to take into account the correlation between risk factors in order to generate realistic stress scenarios. For example, if we were to create a scenario reflecting a sharp devaluation for an emerging markets currency, we would expect to see a snowball effect causing other currencies in the region to lose value as well.

Given the importance of including expert views on stress events and accounting for potential changes in every risk factor, we need to come up with user-defined scenarios for every single variable affecting the value of the portfolio. To facilitate the generation of these comprehensive user-defined scenarios, we have developed a framework in which we can express expert views by defining changes for a subset of risk factors (core factors), and then make predictions for the rest of the factors (peripheral factors) based on the user-defined variables. The predictions for changes in the peripheral factors correspond to their expected change, given the changes specified for the core factors.

What does applying this method mean? If the core factors take on the user-specified values, then the values for the peripheral risk factors will follow accordingly. Intuitively, if the user specifies that the three-month interest rate will increase by ten basis points, then the highly correlated two-year interest rate would have an increase equivalent to its average change on the days when the three-month rate went up by ten basis points.

For example, let us say that we have invested USD 1,000 in the Indonesian JSE equity index, and are interested in the potential scenario of a 10% IDR devaluation. Instead of explicitly specifying a return scenario for the JSE index, we would like to estimate the potential change in the index as a result of a 10% currency devaluation. In the predictive framework, we would specify the change in the JSE index as

$$\Delta JSE = \mu_{\text{JSE}} + \beta(-.10 - \mu_{\text{IDR}}), \tag{4.5}$$

where ΔJSE denotes the change in the equity index, β denotes the beta of the equity index with respect to the FX rate, and the expected return on the two risk factors (μ_{JSE} and μ_{IDR}) are set to zero.[2] In this example, $\beta = 0.2$, so that if the IDR drops 10%, then the JSE index would decrease on average by 2%.

Equation (4.5) illustrates the method to predict peripheral factors when we have only one core factor. In this case, the predictions are simply based on the beta of the peripheral factor with respect to the core factor. This means that the magnitude of the change in the peripheral factor corresponds to the correlation between the core and the peripheral factor scaled by the ratio of their volatilities.[3]

We can generalize this method to incorporate changes in multiple core factors. We define the predicted returns of the peripheral factors as their conditional expectation given that the returns specified for the core assets are realized. We can write the unconditional distribution of risk factor returns as

$$\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \right), \tag{4.6}$$

where $\mathbf{r}_2$ is a vector of core factor returns, $\mathbf{r}_1$ is the vector of peripheral factor returns, and the covariance matrix has been partitioned. It can be shown that the expectation of the peripheral factors ($\mathbf{r}_1$) conditional on the core factors ($\mathbf{r}_2$) is given by

$$E[\mathbf{r}_1|\mathbf{r}_2] = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{r}_2 - \mu_2). \tag{4.7}$$

Setting $\mu_1 = \mu_2 = 0$ reduces (4.7) to

$$E[\mathbf{r}_1|\mathbf{r}_2] = \Sigma_{12}\Sigma_{22}^{-1}\mathbf{r}_2, \tag{4.8}$$

where Σ_{12} is the covariance matrix between core and peripheral factors, and Σ_{22} is the covariance matrix of the core risk factors.[4]

Equation (4.8) is also useful outside of a predictive stress testing environment. Some readers might already have recognized that it corresponds to a multivariate regression, and hence can be used to extract information about the sensitivity of a price series with respect to a set of factors. For example, we could analyze the sensitivity of Coca-Cola stock to a one standard deviation move in a beverage sector index.

[2] Beta is defined as $\beta = \rho\sigma_{\text{JSE}}/\sigma_{\text{IDR}}$, where ρ is the correlation between JSE and IDR returns.

[3] We make the assumption that the correlations and volatilities do not change as a result of the change in the core factor. Finger and Kim (2000) derive a method to introduce correlation breakdowns in stress tests.

[4] Kupiec (1998) uses the conditional distribution (mean and covariance) of peripheral factors, given core factor returns, to compute VaR under stress scenarios.

Example 4.3 Predictive stress test: a currency crisis

We can continue with Example 4.2 and analyze the effect that the currency devaluations would have on the equity positions. Based on (4.8), we need the covariance matrix of risk factor returns. Table 4.4 shows the covariance between the risk factors for our portfolio. The covariances between the core and peripheral factors (Σ_{12}) correspond to the upper right quadrant of the table, the covariances between the core factors (Σ_{22}) correspond to the lower right quadrant, and the upper left quadrant corresponds to the covariance between the peripheral factors (Σ_{11}).

Table 4.4: Covariance matrix of risk factor returns

	Bovespa	JSE	WIG	BRL	IDR	PLN
Bovespa	2.9130	-0.0055	0.2767	0.0360	0.0972	0.2759
JSE	-0.0055	0.9308	0.0769	0.0093	0.2766	-0.0971
WIG	0.2767	0.0769	0.8225	-0.0336	0.0064	0.0900
BRL	0.0360	0.0093	-0.0336	0.2035	-0.0650	0.1309
IDR	0.0972	0.2766	0.0064	-0.0650	1.4070	-0.2123
PLN	0.2759	-0.0971	0.0900	0.1309	-0.2123	0.3633

Using (4.8) and the numbers from Table 4.4, we can calculate the returns on the equity indices conditional on a 10% devaluation on each of the currencies. Table 4.5 shows the predicted returns on the peripheral (equity) factors.

Table 4.5: Logarithmic returns for the peripheral factors

Factor	Return
Bovespa	-8.59%
JSE	-1.83%
WIG	-0.57%

We can finally find the change in value of our portfolio by using the returns from Table 4.5 together with the 10% devaluation scenario for each currency. Table 4.6 contains the results from the stress test.

The result of our stress test shows that the risk of our stress event is USD 95.52 higher than the figure we obtained from the simple stress test. This difference arises from the impact that the currency devaluation has on the equity indices. ■

This chapter concludes the description of the models used to measure the exposure of portfolios to hypothetical stress scenarios.[5] In the previous chapters, we have presented two models for the behavior of risk factor

Table 4.6: Predictive stress test results

Country	Change in Value (USD)
Brazil	-174.09
Indonesia	-116.31
Poland	-105.12
Total	-395.52

returns that work well under normal market conditions. Since normal market conditions are presumed not to hold during stress events, the assumptions embedded in the statistical models of Chapters 2 and 3 might be unrealistic. In light of this inherent weakness, models of return distributions have to be complemented by stress tests.

In Part I we have discussed statistical models that describe the distribution of risk factors, as well as methods to explore potential losses under stress scenarios. Up to this point, we have taken for granted the availability of pricing functions expressed in terms of the underlying risk factors. Part II provides a detailed discussion of a representative set of instruments that illustrates the approaches used to construct pricing functions.

[5] For further reading on stress testing see the April 2000 issue of the RiskMetrics Journal. A list of the articles in the Journal can be found in Appendix B.

Part II

PRICING CONSIDERATIONS

Chapter 5

Pricing Framework

As we point out in Chapter 1, the price of a financial instrument is a function of a set of risk factors. In this chapter, we will explain the pricing methodology RiskMetrics uses to value various financial instruments. The methodology is meant to be used in VaR calculations (through either parametric or simulation methods), stress testing, and other risk management processes. For these applications, especially for VaR estimation, hundreds or thousands of valuations are usually required to obtain the P&L distribution of a complex instrument. Therefore, unlike for the valuation of just a few instruments in trading, we must pay special attention to calculation expenses. The pricing models for these complex instruments are chosen, or, in some cases, constructed so that with minimal sacrifice in accuracy, a great reduction in computational expense can be achieved.

While the pricing methods for a number of financial instruments are included as examples in this chapter, they are used only to illustrate our approaches, and are not meant to give an exhaustive list of the instruments that are covered in RiskMetrics.[1]

5.1 Discounting and mapping cash flows

5.1.1 Basic pricing concepts and conventions

The RiskMetrics building blocks for describing a position are its cash flows. A cash flow is specified by an amount of a currency, and a payment date.

Once determined, these cash flows are marked-to-market. Marking-to-market a position's cash flows means determining the present value of the cash flows given current market rates and prices. The present value of

[1] For a full list of covered instruments contact RiskMetrics (contact information on the back cover).

a cash flow is found by multiplying the cash flow amount by a discount factor, which depends on current market rates, and particularly, the zero-coupon rate on instruments that pay no cash flow until the considered date. It is important that the zero-coupon rates be extracted from instruments which are similar to the position we want to value. For example, in marking-to-market a cash flow from an instrument issued by the U.S. Treasury, Treasury rates will be used, while for a cash flow from a Aa-rated financial corporate bond, the financial corporate Aa zero rate curve will be a good choice if a firm-specific zero rate curve is not available.

To determine the discount factor, we must first lay out the conventions we use to express interest rates. As indicated in Chapter 1, RiskMetrics works with continuously compounded rates. The conversion formula between discretely and continuously compounded rates is given by

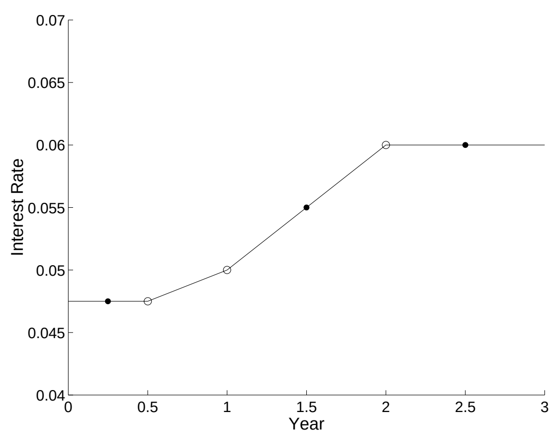
$$z = m \log\left(1 + \frac{z^{(m)}}{m}\right), \tag{5.1}$$

where z is the continuously compounded rate, and $z^{(m)}$ is the discretely compounded annual rate with a compounding frequency of m.

Generally, there is a different interest rate for each future maturity. The relationship between the interest rates and the payment dates of cash flows is called the *term structure* of interest rates.

RiskMetrics treats yield curves as piecewise linear, where points outside of the first and last maturity vertices take on the value of the nearest vertex. As shown in Figure 5.1, suppose a curve is composed of the six months, one year, and two-year rates: $z_{0.5} = 0.0475$, $z_1 = 0.05$, and $z_2 = 0.06$. The three-month rate would be 0.0475, the 1.5-year rate would be 0.055, and the 2.5-year rate would be 0.06.

Figure 5.1: Interpolation of interest rates from term structure



With the above specified conventions, suppose that we have a single cash flow of USD 1 at a future time t, and that the annualized zero-coupon rate for time t is z_t, then the present value of this cash flow is given by $e^{-z_t t}$. By definition, this is also the discount factor for a cash flow at time t.

Example 5.1 Fixed-coupon bond

Let us consider a two year bond with a principal of USD 100 and a fixed coupon rate of 5% per annum. Suppose the coupons are paid semiannually, and the zero-coupon curve is the one shown in Figure 5.1. To calculate the present value of cash flows, we discount the first coupon of USD 2.5 at the six-month zero-coupon rate (4.75%), the second coupon at the one-year zero-coupon rate (5%), and so on. The price of the bond is then given by the sum of all the discounted cash flows:

$$2.5e^{-0.0475\times 0.5} + 2.5e^{-0.05\times 1} + 2.5e^{-0.055\times 1.5} + (2.5+100)e^{-0.06\times 2} = \text{USD } 98.03. \tag{5.2}$$

■

5.1.2 Cash flow mapping for parametric VaR calculations

In the RiskMetrics methodology, a portfolio of financial instruments is broken down into a number of future cash flows. This is adequate for pricing purposes. However, in the parametric VaR calculation, the large number of combinations of cash flow dates leads to the impractical task of computing an intractable number of volatilities and correlations, especially when we are considering a portfolio with many financial instruments. The RiskMetrics methodology simplifies the time structure by mapping each cash flow to a pre-specified set of RiskMetrics vertices. An example set of the vertices is listed below:

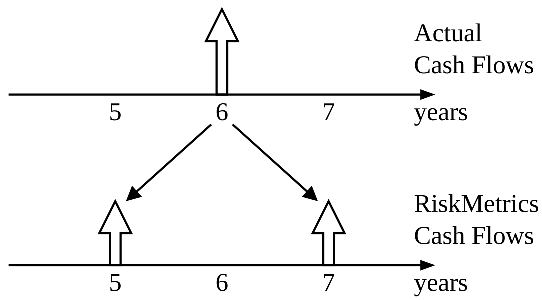
1m 3m 6m 1yr 2yr 3yr 4yr 5yr 7yr 9yr 10yr 15yr 20yr 30yr.

Mapping a cash flow means splitting it between two adjacent RiskMetrics vertices. Figure 5.2 shows how the actual cash flow occurring at six years is split into the synthetic cash flows occurring at the five-year and seven-year vertices. After the cash flow map, a portfolio of instruments are transformed into a portfolio of standard cash flows. Now, for parametric VaR calculations, we only need to take care of volatilities and correlations for these standard vertices. Values of these volatilities and correlations are provided in the RiskMetrics data set.

There is no unique way of splitting a cash flow between two vertices. The original RiskMetrics cash flow map sets the volatility of the considered cash flow to be the linear interpolation of the volatilities on the two neighboring vertices. The cash flow map is then worked out so that this interpolated volatility and the present value of the cash flow are preserved. This approach performs very well under most circumstances. However, it has certain drawbacks. First of all, the volatility in this map does not match the volatility from Monte Carlo simulation, where the interest rate, rather than the volatility itself, is interpolated from the corresponding values on neighboring vertices. In addition, the original cash flow map could produce undesirable results if the correlation between the returns of the zero-coupon bonds corresponding to the two neighboring vertices is very small (though this is a very rare event).[2]

[2] For an example and more details refer to Mina (1999).

Figure 5.2: RiskMetrics cash flow map



The improved RiskMetrics cash flow map starts with the linear interpolation of interest rates, i.e.

$$z_t = \alpha z_L + (1-\alpha) z_R, \tag{5.3}$$

where $\alpha = (t_R - t)/(t_R - t_L)$, t is the maturity of the zero-coupon bond, t_L and t_R are the two adjacent vertices, and z_L and z_R are the two corresponding zero rates at the vertices. Note that this procedure is consistent with the Monte Carlo simulation of VaR for a zero-coupon bond.

To work out the cash flow map, let us first assume that a payment of USD 1 at time t is mapped into a payment of W_L at time t_L, and a payment of W_R at time t_R, as well as a cash position C. To preserve the present value (V_t) of the cash flow, we should have the following equivalence:

$$V_t = e^{-z_t t} = W_L e^{-z_L t_L} + W_R e^{-z_R t_R} + C. \tag{5.4}$$

The cash flow map should also preserve the sensitivity of the present value to changes in the zero rates for the two neighboring vertices. This is equivalent to taking the partial derivative of (5.4) with respect to z_L or z_R while keeping W_R, W_L and C constant. With the expression of z_t in (5.3), we have

$$\frac{\partial V_t}{\partial z_L} = -\alpha t e^{-z_t t} = -W_L t_L e^{-z_L t_L}, \tag{5.5}$$

from which we obtain

$$W_L = \alpha \frac{t}{t_L} e^{-z_t t} e^{z_L t_L}. \tag{5.6}$$

Similarly, we can obtain

$$W_R = (1-\alpha) \frac{t}{t_R} e^{-z_t t} e^{z_R t_R}. \tag{5.7}$$

Finally, from (5.4), (5.6), and (5.7) we can derive the cash position as

$$C = -\frac{(t - t_L)(t_R - t)}{t_R t_L} e^{-z_t t}. \tag{5.8}$$

Therefore, the amount of V_t dollars invested in a zero coupon bond with maturity at time t can be represented by a portfolio of $(\alpha \frac{t}{t_L} V_t)$ dollars (the present value of W_L) invested in a bond with maturity equal to the left vertex, $[(1 - \alpha)\frac{t}{t_R} V_t]$ dollars (the present value of W_R) invested in a bond with maturity equal to the right vertex, and a cash position of $[-\frac{(t-t_L)(t_R-t)}{t_R t_L} V_t]$ dollars.

Example 5.2 Cash flow mapping for a forward rate agreement (FRA)

A forward rate agreement (FRA) is a contract locking in the interest rate that will be applied to a certain notional principal for a specified period in the future.[3]

A typical FRA is stated in terms of the effective date, length of the agreement, forward interest rate, and principal amount. For example, a 3 × 6 FRA locks the interest rate for a three month period between three and six months from the date we entered the contract. FRAs are generally cash-settled at the start of the forward rate agreement period.

In general, a FRA is priced as if one borrows the nominal N at settlement date t_f and pays back the nominal plus interest at a rate r_x at maturity date t_m,

$$Ne^{-z_f t_f} - N[1 + r_x(t_m - t_f)]e^{-z_m t_m} \tag{5.9}$$

where z_f and z_m are the zero rates from the present to t_f and t_m respectively.

Consider a 3 × 6 FRA on EUR one million at a forward rate of 5.136%. Suppose the FRA is entered into on 1-Aug-2000, so the FRA is equivalent to borrowing EUR one million for three months on a discount basis and investing the proceeds for a payoff of $1{,}000{,}000 \times (1 + 0.05136 \times 92/360) =$ EUR 1,013,125 at the end of six months. Here 92 is the number of days between 1-Nov-2000 and 1-Feb-2001.

One month into the trade on 1-Sep-2000, this FRA is now equivalent to borrowing EUR one million for two months on a discount basis and investing the proceeds for the same payoff of EUR 1,013,125 at the end of five months. If we use the example set of RiskMetrics vertices, the two-month cash flow must be mapped to the one-month and three-month vertices, while the five-month cash flow must be split between the three-month and six-month vertices.

The EUR one-, three- and six-month money market rates on 1-Sep-2000 are 4.664%, 4.829% and 5.044% respectively. From linear interpolation, the two- and five-month rates are 4.748%, 4.997%. With this knowledge, the cash flow map for the FRA is shown in Table 5.1.

The parametric VaR can then be calculated based on the volatilities and correlations of the one-month, three-month, and six-month money market rates. ■

[3] For more details on FRAs see Malz (2001a).

Table 5.1: Cash flow map for a FRA

Date	Flow	Term	Yield	PV	Cash	1m	3m	6m
1-Nov-2000	-1,000,000	2m	4.748	-992,127	337,977	-992,127	-337,977	
1-Feb-2001	1,013,125	5m	4.977	992,420	-104,599		519,112	577,907
			Total	293	233,378	-992,127	181,135	577,907

5.2 Generalized framework

5.2.1 Floating rate instruments

Unlike for the simple bond in our first two examples, coupon payments on many financial instruments are not fixed. Their coupon rates are often floating based on certain reference interest rates. Typical examples include floating rate notes and swaps. The pricing of these instruments involves the determination of future values of these interest rates.

Consider a case in which the current zero rates for maturities t_1 and t_2 are z_1 and z_2, respectively. Denote f_{t_1,t_2} as the forward rate for the period of time between t_1 and t_2. The fundamental arbitrage relationship between current and future rates implies that

$$e^{z_1 t_1} e^{f_{t_1,t_2}(t_2-t_1)} = e^{z_2 t_2}. \tag{5.10}$$

The forward rate is solved to be

$$f_{t_1,t_2} = z_1 + \frac{z_2 - z_1}{t_2 - t_1} t_2, \tag{5.11}$$

This implied forward rate will then be used to set cash flows in the pricing of floating rate instruments.

Example 5.3 Floating rate notes (FRN)

Consider an FRN whose coupon rate varies with some interest rate. The floating coupon rate is usually set some time in advance of the actual coupon payment. For example, if coupon payments are paid on a semiannual basis, the current six-month LIBOR rate will be used to determine the payments in six months.

Suppose that we have an FRN that pays six-month LIBOR on a principal of N. Assume the next coupon payment occurs at t_0 with a coupon rate c_0. After that, the coupon payment times are $t_i = t_0 + 0.5i$ ($i = 1, \cdots, n$), and we denote the zero-coupon rates corresponding to these dates by z_i. For the ith coupon payment, the coupon rate is the forward six-month LIBOR rate at t_{i-1}. Using (5.11), we obtain the (continuously compounded) forward six-month LIBOR rate as

$$f_{t_{i-1},t_i} = z_{i-1} + 2(z_i - z_{i-1})t_i. \tag{5.12}$$

Then, using (5.1), we transform this rate into a semiannually compounded rate, which is used as the coupon rate for the ith payment:

$$c_i = 2[e^{0.5 f_{t_{i-1},t_i}} - 1] = 2e^{z_i t_i - z_{i-1} t_{i-1}} - 2, \tag{5.13}$$

where we used the relationship $t_i = t_{i-1} + 0.5$. For example, if the continuously compounded zero rate is 6% for one year, and 6.5% for 1.5 years, then the coupon rate for the payment at 1.5 years will be 7.64%.

The present value of the ith ($i > 0$) coupon payment is given by

$$V_i = N\frac{c_i}{2}e^{-z_i t_i} = Ne^{-z_{i-1}t_{i-1}} - Ne^{-z_i t_i} \tag{5.14}$$

Similar to the fixed coupon bond in our previous example, we can write down the present value of the FRN as the sum of all the discounted cash flows:

$$V = N\frac{c_0}{2}e^{-z_0 t_0} + V_1 + \cdots + V_n + Ne^{-z_n t_n}, \tag{5.15}$$

where the first term on the right hand side is the present value of the next coupon payment, and the last term is the present value of the principal payment at maturity. Using (5.14), the right-hand side is equal to

$$N\frac{c_0}{2}e^{-z_0 t_0} + (Ne^{-z_0 t_0} - Ne^{-z_1 t_1}) + \cdots + (Ne^{-z_{n-1}t_{n-1}} - Ne^{-z_n t_n}) + Ne^{-z_n t_n} \tag{5.16}$$

Equation (5.16) collapses nicely to give the present value of the FRN as:

$$V = N(1 + \frac{c_0}{2})e^{-z_0 t_0}. \tag{5.17}$$

Equation (5.17) implies that immediately after a payment date, a FRN is identical to a newly issued floating-rate note. More importantly, (5.17) implies that the only risk is due to changes in a short term rate. One can easily verify that the same result holds for any compounding frequency. ■

Example 5.4 Plain vanilla interest rate swap (IRS)

An interest rate swap (IRS) is an agreement between two counterparties to exchange fixed for floating cash flows. Although the notional principal is not exchanged in a swap, we can assume without changing the value of the swap that the two counterparties pay each other the same notional amount.[4] It can be thought of as a portfolio consisting of one fixed leg, which is equivalent to a fixed-coupon bond, and one floating leg, which is equivalent to an FRN.

Suppose that a firm enters into an IRS to receive six-month LIBOR and pays 5% per annum semiannually on a notional of USD 100 million. The swap has 1.25 years left to maturity. We also assume that the continuously

[4]This assumption is valid for pricing purposes and market risk analysis, but can result in overstatement of exposure in credit risk analysis.

compounded LIBOR curve is the one shown in Figure 5.1. From Figure 5.1, the continuously compounded zero LIBOR rates for three months, nine months, and fifteen months are interpolated to be 4.75%, 4.875%, and 5.25%. Therefore, the value of the fixed leg is

$$V_{fix} = 2.5e^{-0.0475\times 0.25} + 2.5e^{-0.04875\times 0.75} + (2.5 + 100)e^{-0.0525\times 1.25} = \text{USD } 100.87 \text{ million.} \tag{5.18}$$

Assume that the next coupon rate for the floating leg was set to be 6.0%. Using (5.17), we find the present value of the floating leg to be

$$V_{float} = (100 + 3)e^{-0.0475\times 0.25} = \text{USD } 101.78 \text{ million.} \tag{5.19}$$

Therefore, the swap is worth

$$V = V_{float} - V_{fix} = 101.78 - 100.87 = \text{USD } 0.91 \text{ million} \tag{5.20}$$

■

5.2.2 Expansion of the framework

In the last two examples, the reference curves for setting coupon rates and discounting cash flows were the same. As shown in (5.17), this feature results in a significant simplification of the calculation, and is used in *RiskMetrics Classic* as the basis to price floating rate instruments.

Equation (5.17) is only valid when the reference and discount curves are the same. However, this is not always the case. One exception is a LIBOR-based floating rate note issued by a U.S. corporate. Due to the difference in credit standing, a *future* cash flow of one dollar from a Baa-rated corporate is not worth as much as one dollar from a bank which can borrow at LIBOR. The use of the LIBOR curve for discounting will end up overpricing the FRN. In this case, the right discount curve is the U.S. corporate yield curve which reflects the credit rating of the issuer. This type of floating instrument is incorporated into RiskMetrics by expanding the original pricing framework to allow the specification of different reference and discount curves.

Example 5.5 U.S. corporate FRN

Let us consider a Baa-rated FRN issued by a U.S. corporate. The FRN pays six-month LIBOR and matures in 1.25 years. From Figure 5.1, the zero LIBOR rates for three months, nine months, and fifteen months are found to be 4.75%, 4.875%, and 5.25%. The forward six-month LIBOR rates (after conversion to semiannually compounded rates) are 5.00% from three to nine months, and 5.90% from nine to 15 months. Therefore, the implied coupon rates are 5.00% nine months from now and 5.90% 15 months from now.

From (5.17), if we assume the next coupon rate is 6.0%, then the present value of the FRN will be USD 101.78 (using LIBOR for discounting). Now, suppose the U.S. corporate Baa zero rates for three, nine, and

15 months are 7.0%, 8.4%, and 8.6% respectively. Using the Baa curve for discounting, the price of the FRN is calculated as[5]

$$3e^{-0.070\times0.25} + 2.5e^{-0.084\times0.75} + (2.95+100)e^{-0.086\times1.25} = \text{USD } 97.752, \tag{5.21}$$

which is much lower than the LIBOR-discounted value of USD 101.78. ■

Note that while (5.17) only depends on the short term interest rate, (5.21) depends on the interest rates up to the time of maturity. Therefore, the price sensitivity to longer term interest rates is another important feature that will be missed if one uses the same reference and discount curves in this example. Furthermore, we have sensitivity to both the Baa and LIBOR curves, instead of the LIBOR curve only.

In a generic case, not only can the two curves be different, but also the principal (e.g. amortizing swap), the payment date (e.g. spread-lock), the accrual period (e.g. LIBOR-in-arrears swap), and other elements of cash flows can vary. These variations, along with the associated adjustments (e.g., convexity and timing), are handled by the generalized pricing framework.

Example 5.6 Constant maturity Treasury (CMT) swap

A constant maturity Treasury (CMT) swap is an agreement to exchange LIBOR for a particular constant maturity Treasury rate. A CMT rate is the par yield of a Treasury security with fixed maturity (e.g., five years). In general, there is not a security in the market with the exact desired maturity, and hence the par yield is often interpolated from a specified set of securities (usually the two closest to the desired maturity). In practice, since it is difficult to track the specific issues, we obtain the par yield from a synthetic constant maturity bond constructed from the Treasury yield curve.

As in a plain vanilla interest rate swap, the discount curve for the LIBOR leg and CMT leg is the same USD swap curve. However, while the reference curve for the LIBOR leg is still the USD swap curve, the coupon rate for the CMT leg is computed from the USD Treasury curve.

For illustration, consider a CMT swap paying three-month LIBOR and receiving ten-year CMT rates quarterly on a notional N. Suppose that there are $(n+1)$ payments at times t_i $(i = 0, 1, \cdots, n)$, where t_0 is the next payment date with known coupon rate.

Now, let us focus on the CMT-rate leg of the swap. Note that for this leg, the compounding frequency for the reference curve is two, while the payment frequency is four. For the $(i+1)th$ $(i > 0)$ payment, the coupon rate for the CMT leg is taken as the ten-year CMT rate ($y_{t_i}^{\text{CMT}}$) observed at t_i. The payment for the CMT leg at time t_{i+1} is then

$$N y_{t_i}^{\text{CMT}} \Delta t_i, \tag{5.22}$$

[5]To make the calculation more precise, one has to consider small adjustments to account for the more subtle effects caused by the difference in reference and discounting curves. For more information see Hull (1997).

where $\Delta t_i = t_{i+1} - t_i = 0.25$ is the accrual period for this payment.

Although the future CMT rate $y_{t_i}^{\text{CMT}}$ is not directly observable today, we can use current market information to estimate its *expected* value, an average of all possible future spot CMT rates. The immediate candidate for the future CMT rate is the ten-year CMT *forward* rate ($f_{t_i}^{\text{CMT}}$), that is, the implied par yield for a ten-year Treasury note issued at time t_i, which can be solved from:

$$1 = \sum_{k=1}^{20} \frac{f_{t_i}^{\text{CMT}}}{2} \exp(-f_{t_i,t_i+0.5k}^{\text{Treasury}} \cdot \frac{k}{2}) + \exp(-f_{t_i,t_i+10}^{\text{Treasury}} \cdot 10), \tag{5.23}$$

where the denominator 2 under $f_{t_i}^{\text{CMT}}$ is the compounding frequency for Treasury notes, $f_{t_i,t_i+0.5k}^{\text{Treasury}}$ is the implied forward Treasury zero rate from t_i to $t_i + 0.5k$ ($k = 1, 2, \cdots$), which is in turn obtained from current Treasury zero rates through (5.11).

However, the forward CMT rate is not the same as the expected CMT spot rate.[6] A convexity adjustment is necessary to account for the fact that the expected future yield is higher than the forward yield due to the non-linearity (convexity) of the bond price $G(y)$ at time t_i as a function of its yield (y),

$$G(y) = \sum_{i=1}^{20} \frac{f_{t_i}^{\text{CMT}}/2}{(1+y/2)^i} + \frac{100}{(1+y/2)^{20}}. \tag{5.24}$$

Another adjustment, called the timing adjustment, is also necessary to account for the fact that the rate is set at time t_i based on a reference curve, while the payment is made at time t_{i+1} and is discounted with a different (LIBOR) curve.

Without going into details, the *expected* ten-year CMT rate at time t_{i+1} can be approximated as

$$\overline{y_{t_i}^{\text{CMT}}} = f_{t_i}^{\text{CMT}} + \frac{1}{2}(f_{t_i}^{\text{CMT}})^2(\sigma_{\text{CMT}})^2 \frac{G''(f_{t_i}^{\text{CMT}})}{|G'(f_{t_i}^{\text{CMT}})|} t_i - \frac{f_{t_i}^{\text{CMT}} f_{t_i,t_{i+1}}^{\text{LIBOR}} \sigma_{\text{CMT}} \sigma_L \rho \Delta t_i}{1 + f_{t_i,t_{i+1}}^{\text{LIBOR}} \Delta t_i} t_i, \tag{5.25}$$

where $f_{t_i,t_{i+1}}^{\text{LIBOR}}$ is the implied forward three-month LIBOR rate between time t_i and t_{i+1}, σ_{CMT} is the volatility of the forward CMT rate which is implied from bond options, σ_L is the volatility of the forward LIBOR rate which is implied from caplet prices, and ρ is the correlation between the forward CMT rate and forward LIBOR rate. The second term on the right hand side of (5.25) is the convexity adjustment, while the third term is the timing adjustment.

The expected payoff for the CMT leg at time t_{i+1} will then be $N\overline{y_{t_i}^{\text{CMT}}}\Delta t_i$. With the LIBOR leg priced as a normal FRN according to (5.17), the present value of the swap is

$$N \sum_{i=0}^{n} \overline{y_{t_i}^{\text{CMT}}} \Delta t_i e^{-z_i^L t_i} - N(1 + c_0 \Delta t_0) e^{-z_0^L t_0}, \tag{5.26}$$

where z_i^L is the current zero swap rate for time t_i. ■

[6] See, for example, Hull (1997).

5.3 Equity, foreign exchange and commodities

In this section, we will show how RiskMetrics prices other basic financial instruments related to equities, foreign exchange, commodities, and futures.

Example 5.7 Equity futures

Because the maintenance margins are marked-to-market daily, equity futures contracts are in effect closed out and rewritten at new prices each day. Therefore, the daily gain or loss on a futures contract is the daily change in the futures price, which means the present value of an equity futures position is

$$N(C - F_0). \tag{5.27}$$

where N is the number of commodity units, C is the quoted futures price of the equity, and F_0 is the futures price at the time we entered the contract. If the quoted price C is not available, the following formula can be used for the present value of an equity futures position

$$NSe^{(z_m - q)t_m} - NF_0, \tag{5.28}$$

where z_m is the discount rate for expiration time t_m, q is the continuous dividend rate, F_0 is the entry price, and N is the contract size. Note that the first term in (5.28) is the current theoretical futures price.

Consider a long position of December (2000) S&P 500 index futures with an entry price of 1,400. On 1-Aug-2000, the S&P 500 (cash) index is 1,438.10. There are 135 days between 1-Aug-2000 and the expiration day, 14-Dec-2000. Suppose the S&P 500 dividend yield is 1.1%, the financing costs or interest rates are 6.72% for three months, and 6.895% for six months. The interest rate for 135 days is interpolated to be 6.80%. Then the cash value of the futures contract is

$$\text{USD } 250 \times (1{,}438.10e^{(0.068-0.011)\times 135/365} - 1{,}400) = \text{USD } 17{,}185, \tag{5.29}$$

where USD 250 is the multiplier for S&P 500 index futures contracts. ■

Example 5.8 FX forward

FX forwards are modeled as two discount bonds. The value of an FX forward is

$$N_1 S_1^{FX} e^{-z_1 t_m} - N_2 S_2^{FX} e^{-z_2 t_m}, \tag{5.30}$$

where N_1 is the amount of foreign currency one that will be received with current exchange rate S_1^{FX} and discount rate z_1, and N_2 is the amount of foreign currency two that will be received with current exchange rate S_2^{FX} and discount rate z_2. The discount rates used are risk free rates or money market rates for the given currencies. ■

Example 5.9 Commodity futures

The present value of a commodity futures contract is

$$N(C - F_0), \tag{5.31}$$

where N is the number of commodity units, C is the quoted futures price of the commodity, and F_0 is the futures price at the time we enter the contract. Note that except for the definition of C, (5.31) and (5.27) are identical. ■

5.4 Nonlinear instruments and derivatives

Linearity and nonlinearity of instruments refer to the dependence of their values on underlying risk factor changes. For example, the value of a FRA, as shown in (5.9), depends on the values of two zero-coupon bonds maturing at t_f and t_m. If we use $B(t)$ to denote the value of a zero-coupon bond maturing at t, then (5.9) can be rewritten as

$$N\ B(t_f) + N[1 + r_x(t_m - t_f)]\ B(t_m). \tag{5.32}$$

Equation (5.32) shows that FRAs are linear instruments with values proportional to the underlying zero-coupon bond values.

In contrast, with the same notation, the value of an equity futures in (5.28) can be rewritten as

$$\frac{NSe^{-qt_m}}{B(t_m)} - NF_0. \tag{5.33}$$

Therefore, an equity futures is a nonlinear instrument as its value is inversely proportional to the value of a zero-coupon bond. Another example of a nonlinear instrument is a foreign equity, whose pricing is discussed in Example 2.1. Let us revisit the pricing formula

$$V = NS^{EQ}S^{FX}, \tag{5.34}$$

where N is the number of shares, S^{EQ} is the market price of the foreign equity, and S^{FX} is the foreign exchange rate. S^{EQ} and S^{FX} are the two involved risk factors. Since the coefficient for each risk factor is not constant, a foreign equity is clearly a nonlinear instrument.

In this section, we will focus on the pricing of more nonlinear financial instruments. Although we have met some of them before, by far the most common features of nonlinear financial instruments have to do with options.

5.4.1 Black-Scholes and option pricing

In 1973, Fischer Black and Myron Scholes published the first successful model for pricing stock options.[7] The model turns a potential guesswork into a scientific and precise process, and the value of a stock option can now be calculated with great accuracy.

The Black-Scholes formula for the price of a European call option on a non-dividend-paying stock, that is, the right to buy a unit of stock at time T for a price X, is

$$c = S_0\Phi(d_1) - Xe^{-rT}\Phi(d_2), \tag{5.35}$$

where Φ is the cumulative normal distribution function, S_0 is the current stock price, X is the option strike price, T is the time until option expiration, r is the risk-free interest rate, and

$$d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \tag{5.36}$$

$$d_2 = \frac{\ln(S_0/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}, \tag{5.37}$$

where σ is the stock price volatility. European put options, the right to sell a unit of stock at time T for X, can be priced as a call (struck at X) less a forward contract (with a forward price of X). This relationship between a call, a put, and a forward is called put-call parity.[8]

One key feature of the Black-Scholes option pricing model is that the option price does not depend on the expected rates of return. Since expected rates of return reflect risk preferences by investors, risk preferences are irrelevant in the pricing formula. Due to an arbitrage argument (see Black and Scholes (1973)), we can assume that investors are risk neutral, and obtain a pricing formula that is still valid even in the presence of risk aversion.

Note that when we use Monte Carlo simulation to gauge the risk of a decrease in value of an option, future scenarios are first generated based on the "real world" distribution of returns (as explained in Chapter 2). Then, the option is repriced on each scenario using the risk neutral distribution.

Example 5.10 Equity option

Consider a European call option on a stock with continuous dividend rate q. The value is

$$c = S_0e^{-qT}\Phi(d_1) - Xe^{-rT}\Phi(d_2), \tag{5.38}$$

[7] See Black and Scholes (1973).
[8] For a discussion of put-call parity see Hull (1997).

where

$$d_1 = \frac{\ln(S_0/X) + (r - q + (\sigma^2/2))T}{\sigma\sqrt{T}} \tag{5.39}$$

$$d_2 = \frac{\ln(S_0/X) + (r - q - (\sigma^2/2))T}{\sigma\sqrt{T}}, \tag{5.40}$$

Now suppose the current stock price is USD 50, the continuous dividend rate is 1%, the exercise price is USD 50, the volatility is 30% per annum, the risk-free interest rate is 7% per annum, and the time to maturity is three months. Then

$$d_1 = \frac{\ln(50/50) + (0.07 - 0.01 + 0.3^2/2) \times 0.25}{0.3\sqrt{0.25}} = 0.175 \tag{5.41}$$

$$d_2 = \frac{\ln(50/50) + (0.07 - 0.01 - 0.3^2/2) \times 0.25}{0.3\sqrt{0.25}} = 0.025, \tag{5.42}$$

and the option price is calculated to be USD 3.35. ■

The one parameter in the Black-Scholes pricing formula that cannot be directly observed is the volatility of the stock price. It is a pricing parameter that can be calibrated from the option price observed in the market. The volatility obtained from the calibration is usually known as implied volatility. More discussion on implied volatility as a risk factor will be covered in Section 5.5 of this chapter.[9]

The Black-Scholes model has been extended to value options on foreign exchange, options on indices, options on futures contracts, and other derivative instruments and contingent claims. It has become such a popular tool that the Black-Scholes volatility is now a common language for option pricing.

5.4.2 Black's model and interest rate derivatives

A further extension of the Black-Scholes model is the Black model, which assumes that at the maturity of the option, the value of the underlying asset is lognormal with its expected value equal to its forward value.[10] The important feature of Black's model is that the geometric Brownian motion assumption for the evolution of the underlying asset price is not required.

Using Black's model, the value of a European call option on a certain underlying asset is given by

$$c = e^{-rT}[F\Phi(d_1) - X\Phi(d_2)], \tag{5.43}$$

[9] See also Malz (2001a) and Malz (2001b).
[10] See Black (1976).

where Φ is the cumulative normal distribution function, F is the forward value of the underlying asset, X is the option strike price, T is the time to expiry, r is the risk-free interest rate, and

$$d_1 = \frac{\ln(F/X) + \sigma^2 T/2}{\sigma\sqrt{T}}, \quad (5.44)$$

$$d_2 = \frac{\ln(F/X) - \sigma^2 T/2}{\sigma\sqrt{T}}, \quad (5.45)$$

where σ is the volatility of F.

Black's model is used to value a wide range of European options, including interest rate derivatives, such as bond options, interest rate caps and floors, and European swaptions.

Example 5.11 Bond option

A bond option is an option to buy or sell a particular bond by a certain date for a certain price. With the lognormal assumption of the bond price at the maturity of the option, Black's model can be used to price a bond option. The bond forward price is given by

$$F = (B_0 - c_T)e^{rT}, \quad (5.46)$$

where B_0 is the current bond price, c_T is the present value of the coupons that will be paid before the option maturity date, and r is the discount rate for time T.

Consider a ten-month European call option on a 9.75-year bond with a face value of USD 1,000. Suppose that the current cash bond price is USD 960, the strike price is USD 1,000 (clean price), the ten-month risk-free rate is 10% per annum, and the annualized volatility of the forward bond price in ten months is 9%. The bond pays a semiannual coupon of 10% and the coupon payments of USD 50 are expected in three months and nine months. We assume that the three-month and nine-month risk-free interest rates are 9.0% and 9.5%, respectively. Therefore, the present value of the coupon payment is

$$50e^{-0.25\times 0.09} + 50e^{-0.75\times 0.095} = 95.45. \quad (5.47)$$

According to (5.46), the bond forward price is given by

$$F = (960 - 95.45)e^{0.10\times\frac{10}{12}} = 939.68 \quad (5.48)$$

From the specification of the contract, we know the rest of the parameters are $X = 1{,}000$, $\sigma = 0.09$, $T = 0.833$, and $r = 0.10$. Using (5.43), we find the option price to be USD 9.49. ■

5.4.3 Term structure model and interest rate trees

Although Black's model can be used to value a wide range of instruments, it does not provide a description of the stochastic evolution of interest rates and bond prices. Consequently, for interest rate derivatives whose value depends on the exact paths of such stochastic processes, such as American-style swaptions, callable bonds, and structured notes, Black's model cannot be used. To value these instruments, RiskMetrics employs models which describe the evolution of the yield curve. These models are known as term structure models.

One group of term structure models for instruments pricing are based on no-arbitrage arguments. This group of models are consistent with today's term structure of interest rates observed in the market, meaning that market prices of bonds are recovered by the models. They take the initial term structure as input and define how it evolves. The Ho-Lee model and Hull-White model are two examples in this group which are also analytically tractable. Lognormal one-factor models, though not analytically tractable, have the advantage that they avoid the possibility of negative interest rates.

In RiskMetrics, the Black-Derman-Toy (BDT) model is used to price some complex interest rate derivatives.[11] The principal reason for using BDT model is its ease of calibration to the current interest rate and the implied term structure of volatility.

The direct application of a BDT model is to construct interest rate trees from which complex instruments can be priced. An interest rate tree is a representation of the stochastic process for the short rate in discrete time steps. The tree is calibrated to current market interest rates and volatilities. Note that in an interest rate tree, the discount rate varies from node to node. As an example, let us use a BDT model to build an interest tree, and use it to price a callable bond.

Example 5.12 BDT tree calibration

A BDT tree is fully described by a vector $r = \{r_{i0}\}$, $(i = 1, \cdots, n)$ whose elements are the lowest values of the short rate at time step i, and by a vector $\sigma = \{\sigma_i\}$, whose elements are the annualized volatilities of the short rate from time step i to time step $i + 1$. Figure 5.3 shows an example of a calibrated BDT tree.

The BDT model in its discrete version can be stated as[12]

$$r_{ij} = r_{i0} e^{2\sigma_i j \sqrt{\Delta t}}, \tag{5.49}$$

where Δt is the time step in years and j $(0, 1, \cdots, i - 1)$ is an index representing the state of the short rate at each time step.

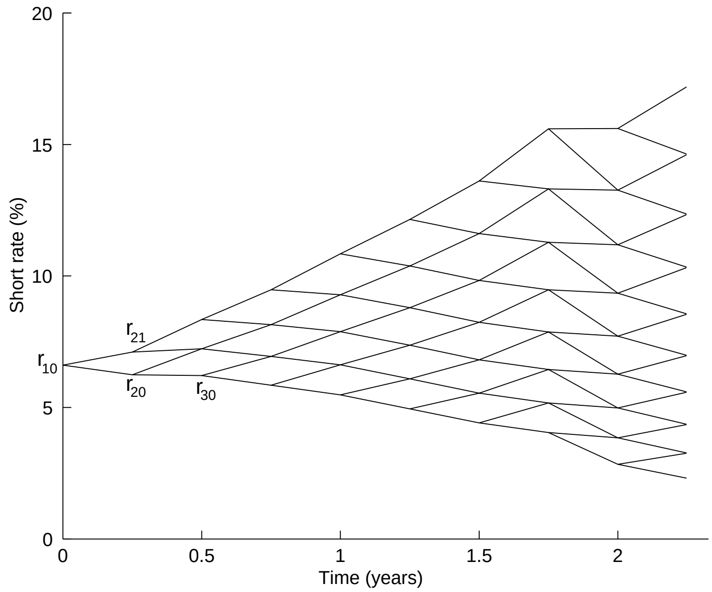
The first step for the calibration is to set each BDT volatility σ_i equal to the implied volatility of a caplet with the same maturity.[13] The short rate vector $r = \{r_{i0}\}$ is then calibrated so that the prices of the BDT

[11] See Black, Derman and Toy (1990).

[12] See Rebonato (1996).

[13] This is not theoretically correct because the forward rates in a BDT tree are only approximately lognormal. However, the induced pricing errors are very small and the time savings in the volatility calibration justify this procedure. For a more detailed discussion of calibration issues see Rebonato (1996).

Figure 5.3: An example of BDT tree



zero-coupon bonds corresponding to each of the nodes in the current yield curve exactly match the true bond prices. To do this, we start with the short maturity of the yield curve and work our way to the longest maturity.

Suppose that we want to build a tree with a step length of three months, or $\Delta t = 0.25$ years. Assume that the market prices for three-month and six-month Treasury bills are USD 98.48 and USD 96.96 respectively, and that the implied volatility of a caplet with expiry in six months is 15%. In the first step we find r_{10} by solving

$$98.48 = \frac{100}{1 + r_{10}\Delta t} = \frac{100}{1 + 0.25 r_{10}}, \tag{5.50}$$

which gives the three-month yield to be $r_{10} = 0.0617$.

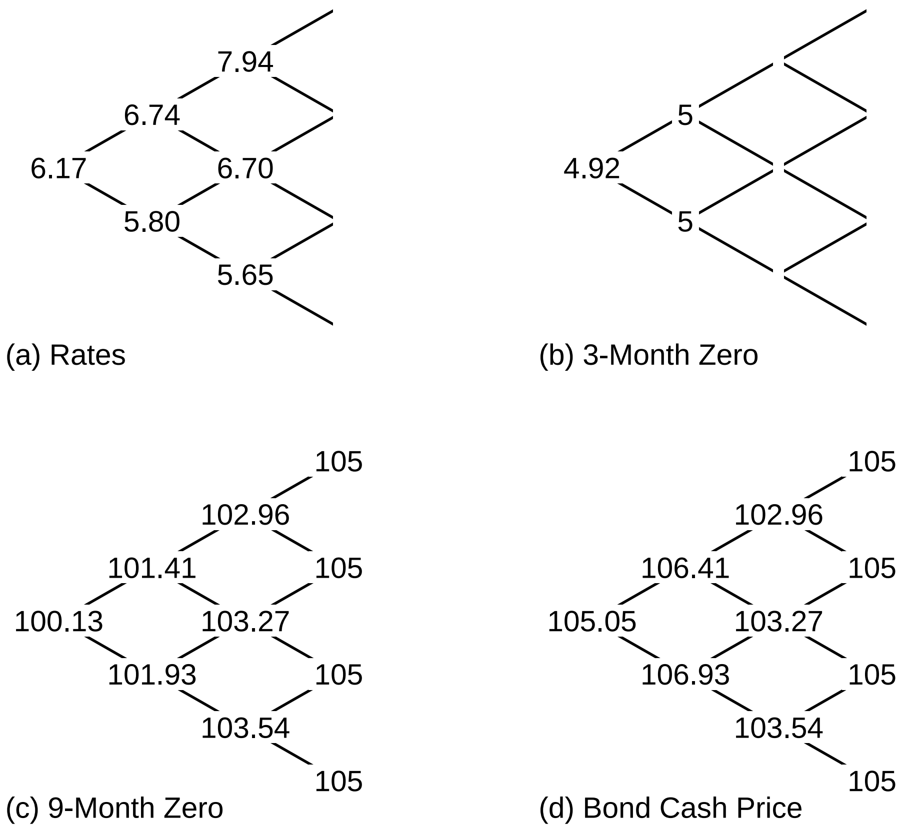
The price for six-month Treasury bills is an average discounted value over all paths out to six months. We have the following relationship for the second step

$$96.96 = \left(\frac{1}{1 + r_{10}\Delta t}\right) \times \frac{1}{2}\left(\frac{1}{1 + r_{20}\Delta t} + \frac{1}{1 + r_{21}\Delta t}\right) \times 100, \tag{5.51}$$

where the first factor on the right hand side of the equation is the discount factor for the first period, and the second factor is the average discount factor for the second period. With the relationship between r_{20} and r_{21} given by (5.49), and the forward volatility $\sigma_2 = 15\%$, r_{20} and r_{21} are found to be 0.0580 and 0.0674 respectively.

If we repeat the process until maturity time of the instrument, we will have a calibrated BDT tree as illustrated in Figure 5.4(a). We can then use it to value fixed income instruments with embedded options, including callable bonds.

Figure 5.4: A coupon bond is valued as a portfolio of zero coupon bonds using a BDT tree.



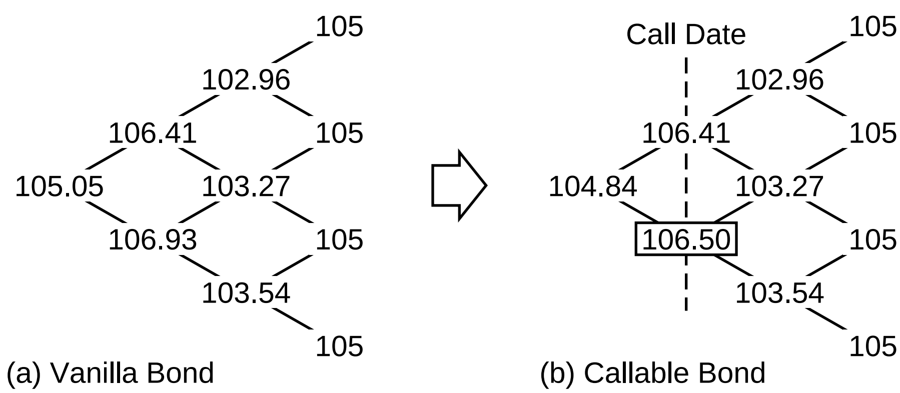
As an illustration of its application, let us use the calibrated BDT tree to price a simple fixed-coupon bond. Consider a bond with a 10% coupon, and nine months left to maturity. It can be represented by a three-month zero coupon bond with a USD 5 face value and a nine-month zero coupon bond with a USD 105 face value. As illustrated in Figure 5.4(b) and (c), we start from the maturity date and discount back cash flows according to the BDT short rates on each node. For example, in Figure 5.4(c), USD 102.96 is obtained by discounting USD 105 by 7.94%, while USD 101.41 is obtained by taking the average of USD 102.96 and USD 103.27 and discounting it by 6.74%. The price of the zero coupon bonds is obtained at the end of the discounting process when we reach time zero. Figure 5.4(d) shows the cash price of the bond is USD 105.05, which is the sum of the three-month zero coupon bond and nine-month zero zero coupon bond prices. ■

Example 5.13 Callable bond

A callable bond is a bond that the issuer can redeem before maturity on a specific date or set of dates (call dates), at specific prices (call prices).

For a callable bond, we need to check in the discounting process if a date corresponding to a node is a call date. If it is a call date and the future bond price on that node is greater than the call price, then we reset that price to the call price before continuing the discounting process.

Figure 5.5: Valuation of a callable bond with BDT tree



Consider a callable bond with the same specification as the plain vanilla bond in Example 5.12, except for a provision which allows the issuer to call the bond three months from now at a call price (clean price) of USD 101.50. We first need to convert this call price into a dirty price by adding the accrued interest of 5 USD (the full coupon because it falls on a coupon date), which means that the dirty price is USD 106.50. We then compare this call price to the prices on the three-month nodes. Since one of the prices (USD 106.93) is higher than the call price (USD 106.50), the bond on that node will be called, and as shown in Figure 5.5, this price is replaced by the call price. The cash price of the bond is then found to be USD 104.84. We can also conclude that the embedded call option is worth USD 0.21 today, which is the difference between the price of the plain vanilla and callable bonds. ■

In the parametric VaR calculation, where the derivatives of the price with respect to its underlying risk factors are required, trees can be used to compute these derivatives for complex instruments. In this case, a tree is first calibrated to current market condition and the price of the considered instrument is computed from this tree. Then a second tree is calibrated to a scenario with a small change in the risk factor. A new price of the instrument is computed from the second tree. Finally, the derivative is taken as the ratio of the price difference and the change of the risk factor.

Note that in Monte Carlo simulation, which involves the construction of interest rate trees, future scenarios of the term structure are generated using the methods described in Chapter 2. Then, in each scenario, the trees must be calibrated again to the generated term structure.

5.4.4 Analytic approximations

With the analytic methodology we covered so far, most options can be valued in closed form. For some of the exceptions whose exact analytic pricing formulas are not available, such as American options and average rate options, closed form pricing formulas can still be obtained after some analytic approximations.

Example 5.14 Average Price Option

An average price call option is a type of Asian option whose payoff is $\max(0, \bar{S} - X)$, where $\bar{S}$ is the average value of the underlying asset calculated over the life of the option, and X is the option strike price. If the underlying asset price S is assumed, as usual, to be lognormally distributed, the distribution of the arithmetic average $\bar{S}$ does not have an analytic form. However, the distribution is found to be approximately lognormal and good precision can be achieved by calculating the first two moments of $\bar{S}$ and using the standard risk-neutral valuation.[14] With the assumption that $\bar{S}$ is lognormally distributed, we can approximate the call on $\bar{S}$ as a call option on a futures contract.

Note that there is more than one method for approximating the price of an arithmetic average price option. For example, Curran's approximation adds accuracy and flexibility in handling multiple averaging frequencies.[15]
■

In Monte Carlo VaR calculations for instruments that have costly pricing functions, such as the callable bonds discussed above, quadratic approximations can be applied to vastly improve the performance. The standard simplification of complicated pricing functions is to approximate them by a second order Taylor expansion. However, Taylor series expansions are only accurate for small changes in the risk factors, and since VaR is concerned with large moves, the use of such approximations can lead to very large errors. The new method is instead based on fitting a quadratic function to the true pricing function for a wide range of underlying risk factor values. This approximation fits the range of interest rates using least square fitting techniques, and ensures a high accuracy for VaR calculations based on the approximate pricing function.

Example 5.15 Quadratic approximation of callable bonds

The valuation of complex derivative exposures can be a computationally intensive task. For risk management purposes, and particularly for VaR, we need hundreds of valuations to obtain the P&L distribution of a complex instrument. It is often the case that closed form solutions do not exist, forcing the use of more expensive numerical methods such as Monte Carlo simulation and finite difference schemes. As shown in Example 5.13,

[14] See Turnbull and Wakeman (1991).
[15] See Curran (1992).

a callable bond can be priced with a calibrated BDT tree. While BDT trees are very convenient for pricing purposes, they are not very practical for VaR calculations.

As we know, callable bond prices depend on the entire term structure of interest rates, and to first order, changes in the term structure are mostly captured by parallel shifts. As a simple illustration of quadratic approximations, the parallel shift can be used as the only risk factor to describe the dynamics of the callable bonds. The scenarios are then chosen with proper combination of the movements (in number of standard deviation) of the risk factor. For example, if we sample at -3, -2, -1, 0, 1, 2, 3 standard deviations for the risk factor, we need only to perform seven full valuations. The quadratic approximation is then fit to these prices using least squares. Each subsequent evaluation for any realization of the risk factor will correspond to a simple matrix multiplication. On the other hand, if we use full revaluation on each scenario, we would need to calibrate the tree each time and then use the calibrated tree to obtain a new bond price.

We can divide the total cost of the full valuation and quadratic approximation methods in two parts. The first is a fixed cost, and the second is the cost per simulation. Full valuation methods have no fixed cost and a large cost per simulation (calibration of the tree and pricing of the bond), while quadratic approximations have a modest fixed cost (seven full valuations plus the least squares to fit the quadratic function) and a very small cost per simulation (a matrix multiplication). Therefore, as the number of simulations increases, the time savings from using the quadratic approximation increases. In addition, if the pricing function is smooth enough, the lost in accuracy from using a quadratic approximation is very small.[16] ■

5.5 Price calibration

Throughout this chapter, from the simple fixed cash flow instruments, to the more complex derivatives, the emphasis has been on providing pricing functions that accurately capture the sensitivity to risk factor movements. Provided that the risk factors we have identified from the outset are sufficient to describe price changes in the instrument in question, this approach guarantees that we obtain the precise distribution of these changes. However, given the current level of risk factors, the pricing functions will often produce results inconsistent with market price levels. Thus, although the distribution of instrument price changes will be accurate, the distribution will be anchored to the wrong current price level.[17] To help approximate market prices, we introduce the notion of price calibration.

The idea behind price calibration is to adjust some pricing parameters to ensure that the current price of an instrument obtained through the pricing function will be consistent with the price observed in the market. The value of this pricing parameter is then held constant during the Monte Carlo simulations. In this section, we present two examples of price calibration: a corporate bond and an equity option. For the bond, we calibrate a spread over a base discount curve to match an observed market price. For the option, we calibrate the implied volatility in the pricing function to match an observed option premium.

[16] For more details about quadratic approximations refer to Mina (2000).

[17] For non-linear instruments, even the shape of the distribution will be different due to the non-linear effects.

Example 5.16 Bond spread calibration

Suppose we have a corporate bond with the same coupon and maturity as the bond in Example 5.1. The Treasury zero-coupon curve is also the same, as shown in Figure 5.1. We know the quoted (clean) market price of the bond is USD 90. Assuming a parallel spread s over the given zero-coupon curve, the pricing formula in Example 5.1 can be rewritten as:

$$90 = 2.5e^{-(0.0475+s)\times 0.5} + 2.5e^{-(0.05+s)\times 1} + 2.5e^{-(0.055+s)\times 1.5} + (2.5+100)e^{-(0.06+s)\times 2}. \tag{5.52}$$

After solving the above equation numerically, the spread is found to be 4.44%. ■

Example 5.17 Implied volatility calibration

Suppose we have an equity call option as specified in Example 5.10. Let us assume that we do not know the volatility, but we know that its market price is USD 3. We can then numerically invert the option pricing equation and back out the implied volatility as 26.47% per annum. ■

It is important to note that since the calibrated spreads and implied volatilities are held constant in Monte Carlo simulation, no extra risks associated with the spreads or volatilities are introduced. Usually, data questions preclude the use of implied volatilities as risk factors. However, considerable progress has been made in this area. For further discussion refer to Malz (2001b).

Part III

STATISTICS AND REPORTS

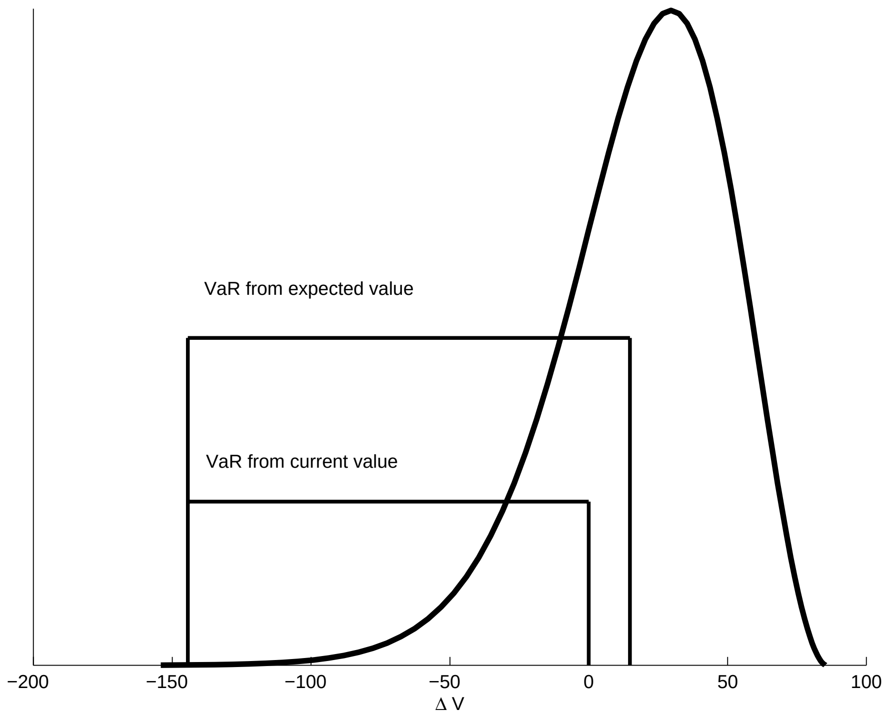
Chapter 6

Statistics

In this chapter we describe a number of risk statistics and explain their calculation through the simulation and parametric methods described in Chapters 2 and 3. It is important to note that our list of statistics is not exhaustive, but rather illustrative of some risk measures widely used in practice. We emphasize that since every statistic has shortcomings, prudent risk management calls for the use of more than one statistic in portfolio risk analysis. This analysis should be analogous to the description of other types of distributions arising in different disciplines, where the mean, median, standard deviation, and percentiles give complementary information about the shape of the distribution. In addition, we also need statistics to help us understand the interaction between market variables. These statistics are analogous to the ones used in multivariate statistical analysis.

VaR is widely perceived as a useful and valuable measure of total risk that has been used for internal risk management as well as to satisfy regulatory requirements. In this chapter, we define VaR and explain its calculation using three different methodologies: closed-form parametric solution, Monte Carlo simulation, and historical simulation. However, in order to obtain a complete picture of risk, and introduce risk measures in the decision making process, we need to use additional statistics reflecting the interaction of the different pieces (positions, desks, business units) that lead to the total risk of the portfolio, as well as potential changes in risk due to changes in the composition of the portfolio. Marginal and Incremental VaR are related risk measures that can shed light on the interaction of different pieces of a portfolio. We will also explain some of the shortcomings of VaR and introduce a family of “coherent” risk measures—including Expected Shortfall—that fixes those problems. Finally, we present a section on risk statistics that measure underperformance relative to a benchmark. These relative risk statistics are of particular interest to asset managers.

Figure 6.1: **Defining VaR**



6.1 Value at Risk

Value-at-Risk (VaR) is one of the most important and widely used statistics that measure the potential risk of economic losses. With VaR, financial institutions can get a sense for the likelihood that a loss greater than a certain amount would be realized. In particular, VaR answers the question: What is the minimum amount that I can expect to lose with a certain probability over a given horizon? For example, VaR can tell us that one out of 20 days I can expect to realize a loss of at least 2% of the total value of my portfolio. In mathematical terms, VaR corresponds to a percentile of the distribution of portfolio P&L, and can be expressed either as a potential loss from the current value of the portfolio, or as the loss from the expected value at the horizon. Figure 6.1 shows the portfolio P&L distribution. The expected P&L is GBP 15 and the first percentile is GBP -140. Hence, we can either express 99% VaR as a loss from the current value (VaR = GBP 140), or as a loss from the expected value (VaR = GBP 155). The decision to anchor the VaR calculation at the current value or the expected value is arbitrary, and for the purposes of this document, we define VaR as the difference between the corresponding percentile of the P&L distribution and the current value of the portfolio.

In this section we show how to calculate VaR using the tools developed in Part I. VaR estimates will generally be different depending on whether we use the parametric method, Monte Carlo simulation, or historical simulation. However, as we will explain in 6.1.1, the methods used to calculate VaR for Monte Carlo and historical simulation are identical once the scenarios have been generated. In this sense, all simulation methods are the same; the difference lies only in the assumptions used to generate scenarios.

6.1.1 Using simulation methods

In Part I we show how to obtain P&L scenarios for a portfolio using Monte Carlo and historical simulation. In this section we show how one can use the generated scenarios to obtain risk measures. It is important to keep in mind that the methodology used to generate scenarios does not make a difference in the calculation of risk measures. In other words, once we have a set of scenarios, we can ignore whether they came from Monte Carlo or historical simulation, and use the same procedure to calculate statistics. In the rest of this section, we will not differentiate between methods to calculate statistics for the two simulation methodologies.

We can calculate VaR numbers using the simulated P&L values. Suppose that we have generated 1,000 P&L scenarios, and that we want to calculate 95% VaR. In this case, since 95% VaR is defined as the fifth percentile of the losses, we would simply calculate VaR as the 50th largest loss scenario.

In general, if we have generated m P&L scenarios, and want to calculate VaR at an α confidence level, we can sort the m P&L scenarios in descending order, denote them by $\Delta V_{(1)}, \Delta V_{(2)}, \ldots, \Delta V_{(m)}$, and define VaR as

$$\text{VaR} = -\Delta V_{(k)}, \tag{6.1}$$

where $k = m\alpha$.[1]

For example, if the generated P&L scenarios (in EUR) sorted in descending order are

$\Delta V_{(1)}$	$\Delta V_{(2)}$	$\cdots$	$\Delta V_{(932)}$	$\cdots$	$\Delta V_{(950)}$	$\cdots$	$\Delta V_{(968)}$	$\cdots$	$\Delta V_{(999)}$	$\Delta V_{(1000)}$
1250	1200	$\cdots$	-850	$\cdots$	-865	$\cdots$	-875	$\cdots$	-950	-1100

(6.2)

then the 95% VaR is $-\Delta V_{(950)} = \text{EUR } 865$.

Confidence intervals

After describing how to calculate VaR using simulation methods, it is important to reflect upon the random nature of the quantity we have estimated (e.g., if we calculated VaR using two different sets of random

[1] The ordered scenarios $V_{(i)}$ are also called order statistics.

scenarios, we would obtain different VaR numbers). Assuming that the model used to generate scenarios is correct, how can one decide how close our number is to the true quantity we are trying to estimate?

Let us try to gain some intuition with a simple example. If we were trying to assess whether or not a coin is fair by flipping it 100 times and counting the proportion of heads that we got, we would observe that we seldom obtain the same number of heads and tails even when the coin is indeed fair. Let us say that we observe 36 heads. Do we have enough evidence to infer that the coin is not fair? One way of answering that question is to construct a confidence interval around the number of heads that we would get with a certain probability if the coin was fair. In our example, we can say that with a probability of 99%, we would observe between 37 and 63 heads if the coin was fair. In other words, there is only a 1% probability that we would observe fewer than 37 or more than 63 heads. Based on the evidence, we can say that the coin is not likely to be fair. As the number of trials increases, we expect to obtain a better sense of whether or not the coin is fair. For example, let us say that we now flip 1,000 times, and that we get 450 heads. What can we say here? The proportion of heads obtained is 45%, much higher than the 36% obtained with 100 flips. However, as the number of trials grows, our estimation error gets smaller, and now our 99% confidence interval is between 460 and 540 heads. In this case, we will also reject the hypothesis that the coin is fair.

Similarly, since we cannot run an infinite number of simulations, our VaR estimates are likely to contain some simulation error. One can calculate the confidence intervals around VaR. These confidence intervals can be used to determine the number of simulations that we want to run. Since there is a tradeoff between accuracy and computing time, we want to run the smallest number of simulations such that we are comfortable with the width of the confidence interval.

We can express a $(1-p)$ confidence interval around VaR in terms of the P&L order statistics. That is, we can say that VaR is between $\Delta V_{(r)}$ and $\Delta V_{(s)}$ with probability $(1-p)$, where

$$r = m\alpha + \sqrt{m\alpha(1-\alpha)}z_{\frac{p}{2}}, \text{ and } \quad s = m\alpha - \sqrt{m\alpha(1-\alpha)}z_{\frac{p}{2}}, \tag{6.3}$$

and $z_{\frac{p}{2}}$ is the corresponding percentile of the standard normal distribution (e.g., $z_{\frac{p}{2}} = -1.64$ if $\frac{p}{2} = 0.05$).[2]

For example, if we had run 1,000 simulations as in (6.2), and calculated 95% VaR as $-\Delta V_{(950)} =$ EUR 865, then, with a probability of 99%, we can say that the true value of VaR is between $-\Delta V_{(932)} =$ EUR 850 and $-\Delta V_{(968)} =$ EUR 875. This means that we can expect the error in our VaR estimate to be around EUR 25. If we are not comfortable with this error size, we can increase the number of simulations to reduce the width of the confidence interval.

6.1.2 Using parametric methods

Based on the analysis from Section 2.3, we can compute risk statistics using parametric methods. An important observation is that the average P&L when we use the parametric method is always equal to zero.

[2] For a discussion of how confidence intervals are derived see Gupton, Finger and Bhatia (1997) Appendix B.

This property comes from the assumption that the relationship between the risk factors and the instrument prices is linear.[3]

To calculate VaR using the parametric approach, we simply note that VaR is a percentile of the P&L distribution, and that percentiles of the normal distribution are always multiples of the standard deviation. Hence, we can use (2.31) to compute the T-day (1 - α)% VaR as

$$\mathrm{VaR} = -z_\alpha \sqrt{T\delta^\top \Sigma \delta}, \tag{6.4}$$

where z_α is the corresponding percentile of the standard normal distribution.

6.2 Marginal VaR

The Marginal VaR of a position with respect to a portfolio can be thought of as the amount of risk that the position is adding to the portfolio. In other words, Marginal VaR tells us how the VaR of our portfolio would change if we sold a specific position. Marginal VaR can be formally defined as the difference between the VaR of the total portfolio and the VaR of the portfolio without the position:

$$\text{Marginal VaR for a position} = \text{VaR of the total portfolio} - \text{VaR of the portfolio without the position.} \tag{6.5}$$

According to this definition, Marginal VaR will depend on the correlation of the position with the rest of the portfolio. For example, using the parametric approach, we can calculate the Marginal VaR of a position p with respect to portfolio P as:

$$\begin{aligned}\mathrm{VaR}(P) - \mathrm{VaR}(P-p) &= \sqrt{\mathrm{VaR}^2(P-p) + \mathrm{VaR}^2(p) + 2\rho\mathrm{VaR}(P-p)\mathrm{VaR}(p)} - \mathrm{VaR}(P-p)\\ &= \mathrm{VaR}(p)\frac{1}{\xi}\left(\sqrt{\xi^2 + 2\rho\xi + 1} - 1\right),\end{aligned}$$

where ρ is the correlation between the position p and the rest of the portfolio $P-p$, and $\xi = \mathrm{VaR}(p)/\mathrm{VaR}(P-p)$ is the ratio of the VaR of the position to the VaR of the rest of the portfolio.

Note that Marginal VaR is an increasing function of the correlation between the position and the portfolio. Marginal VaR will be positive when $\rho \geq 0$, and negative when $\rho < 0$. In addition, when the VaR of the position is much smaller than the VaR of the portfolio, Marginal VaR is approximately equal to the VaR of the position times ρ. That is,

$$\text{Marginal VaR} \to \mathrm{VaR}(p)\rho \quad \text{as} \quad \xi \to 0. \tag{6.6}$$

[3]The only way to obtain an average P&L different from zero under our model for factor returns, is to use a non-linear pricing function of risk factors. This is done through Monte Carlo simulation.

To get some intuition about Marginal VaR we will examine three extreme cases:

1. If $\rho = 1$, then the position behaves exactly like the portfolio and hence its contribution to the risk of the portfolio is equal to the stand-alone VaR (VaR(p)).

2. If $\rho = -1$, the position behaves as the exact opposite of the portfolio, and thus decreases the risk of the portfolio by an amount equal to its stand-alone VaR.

3. If the portfolio and the position are uncorrelated ($\rho = 0$), then the contribution to the total risk by the position is always positive and equal to $\mathrm{VaR}(p)(\sqrt{1+\xi^2}-1)/\xi$. Equation (6.6) implies that if we add a small uncorrelated position to a portfolio the increase in risk will be negligible.

In a similar fashion, we can compute the Marginal VaR of a group of positions with respect to a portfolio. For example, we could use Marginal VaR to identify the amount of risk contributed by each specific desk to the global portfolio of an investment bank, or the amount of risk added by all the positions denominated in a foreign currency.

6.3 Incremental VaR

In the previous section we explained how Marginal VaR can be used to compute the amount of risk added by a position or a group of positions to the total risk of the portfolio. However, we are also interested in the potential effect that buying or selling a relatively small portion of a position would have on the overall risk. For example, in the process of rebalancing a portfolio, we often wish to decrease our holdings by a small amount rather than liquidate the entire position. Since Marginal VaR can only consider the effect of selling the whole position, it would be an inappropriate measure of risk contribution for this example.

Incremental VaR (IVaR) is a statistic that provides information regarding the sensitivity of VaR to changes in the portfolio holdings. For example, if we have a portfolio of equities containing a position of USD 1,000 in IBM stock, and the IVaR of IBM within our portfolio is USD 100, we can say that if we increase our holdings in IBM to USD 1,100, the VaR of our portfolio would increase by approximately (1,100 / 1,000 - 1) $\times$ USD 100 = USD 10. In a similar way, if we denote by IVaR_i the Incremental VaR for each position in the portfolio, and by θ_i the percentage change in size of each position, we can approximate the change in VaR by

$$\Delta\mathrm{VaR} = \sum \theta_i \mathrm{IVaR}_i. \tag{6.7}$$

Another important difference between IVaR and Marginal VaR is that the sum of the IVaRs of the positions add up to the total VaR of the portfolio. In other words, $\sum \mathrm{IVaR}_i = \mathrm{VaR}$. This additive property of IVaR has important applications in the allocation of risk to different units (desks, sectors, countries), where the goal is to keep the sum of the risks equal to the total risk.

To explain the calculation of IVaR, we need a more rigorous definition. Let w_i be the amount of money invested in instrument i. We define the Incremental VaR of instrument i as

$$\text{IVaR}_i = w_i \frac{\partial \text{VaR}}{\partial w_i}. \tag{6.8}$$

To verify that $\sum \text{IVaR}_i = \text{VaR}$ we need to note that VaR is a homogeneous function of order one of the total amount invested. This means that if we double the investments on each position, the VaR on the new portfolio will be twice as large. That is, $\text{VaR}(tw_1, tw_2, \ldots, tw_n) = t\text{VaR}(w_1, w_2, \ldots, w_n)$. Then, by Euler's homogeneous function theorem we have that $\text{VaR} = \sum w_i \frac{\partial \text{VaR}}{\partial w_i}$.

Calculation of IVaR

Parametric methods

Let us assume that we have an equity portfolio and want to calculate the Incremental VaR of each position. Following (6.8), we need to evaluate the derivative of the VaR of the portfolio with respect to the size of each position. Since the size of a position in equities (in currency terms) is equal to the delta equivalent for the position, we can express the VaR of the portfolio in (6.4) as

$$\text{VaR} = -z_\alpha \sqrt{w^\top \Sigma w}. \tag{6.9}$$

We can then calculate IVaR for the i-th position as:

$$\begin{aligned}
\text{IVaR}_i &= w_i \frac{\partial \text{VaR}}{\partial w_i} && (6.10)\\
&= w_i \left(-z_\alpha \frac{\partial \sqrt{w^\top \Sigma w}}{\partial w_i} \right) && (6.11)\\
&= w_i \left(-\frac{z_\alpha}{\sqrt{w^\top \Sigma w}} \sum_j w_j \Sigma_{ij} \right). && (6.12)
\end{aligned}$$

Hence,

$$\text{IVaR}_i = w_i \nabla_i, \tag{6.13}$$

where

$$\nabla = -z_\alpha \frac{\Sigma w}{\sqrt{w^\top \Sigma w}}. \tag{6.14}$$

The vector ∇ can be interpreted as the gradient of sensitivities of VaR with respect to the risk factors. Therefore, (6.13) has a clear interpretation as the product of the exposures of the position with respect to each risk factor (w_i), and the sensitivity of the VaR of the portfolio with respect to changes in each of those risk factors (∇).

Simulation methods

The parametric method described above produces exact results for linear positions such as equities. However, if the positions in our portfolio are not exactly linear, we need to use simulation methods to compute an exact IVaR figure.

We might be tempted to compute IVaR as a numerical derivative of VaR using a predefined set of scenarios and shifting the investments on each instrument by a small amount. While this method is correct in theory, in practice the simulation error is usually too large to permit a stable estimate of IVaR. In light of this problem, we will use a different approach to calculate IVaR using simulation methods. Our method is based on the fact that we can write IVaR in terms of a conditional expectation. To gain some intuition, let us say that we have calculated VaR using Monte Carlo simulation. Table 6.1 shows a few of the position scenarios corresponding to the portfolio P&L scenarios in (6.2). In our example, since we have 1,000 simulations, the 95% VaR corresponds to the 950th ordered P&L scenario ($-\Delta V_{(950)} =$ EUR 865). Note that VaR is the sum of the P&L for each position on the 950th scenario. Now, if we increase our holdings in one of the positions by a small amount while keeping the rest constant, the resulting portfolio P&L will still be the 950th largest scenario and hence will still correspond to VaR. In other words, changing the weight of one position by a small amount will not change the order of the scenarios. Therefore, the change in VaR given a small change of size h in position i is $\Delta\text{VaR} = hx_i$, where x_i is the P&L of the position in the 950th scenario. Assuming that VaR is realized only in the 950th scenario we can write:

$$w_i \frac{\partial \text{VaR}}{\partial w_i} = \lim_{h\to 0} w_i \frac{hx_i}{h} \tag{6.15}$$

$$= w_i x_i. \tag{6.16}$$

We can then make a loose interpretation of Incremental VaR for a position as the position P&L in the scenario corresponding to the portfolio VaR estimate. The Incremental VaR for the first position in the portfolio would then be roughly equal to EUR 31 (its P&L on the 950th scenario).

Since VaR is in general realized in more than one scenario, we need to average over all the scenarios where the value of the portfolio is equal to VaR. We can use (6.15) and apply our intuition to derive a formula for IVaR:

$$\text{IVaR}_i = \mathbf{E}[w_i x_i | w^\top x = \text{VaR}]. \tag{6.17}$$

In other words, IVaR_i is the expected P&L of instrument i given that the total P&L of the portfolio is equal to VaR.[4]

While this interpretation of IVaR is rather simple and convenient, there are two caveats. The first is that there is simulation error around the portfolio VaR estimate, and the position scenarios can be sensitive to the choice

[4]Note that this derivation is not rigorous but it can be formalized by imposing some technical conditions on the portfolio distribution function. We have chosen to provide only a heuristic argument.

of portfolio scenario. The second problem arises when we have more than one position in a portfolio leading to more than one scenario that produces the same portfolio P&L. For example, if we have two equities, we can lose EUR 10 if the first equity drops EUR 10 and the second does not move, or if the first remains constant and the second loses EUR 10. Therefore, a more robust estimate of IVaR results from averaging the position P&L scenarios that lead to the portfolio VaR scenario.

Table 6.1: Incremental VaR as a conditional expectation

Scenario #	1	2	⋯	932	⋯	950	⋯	968	⋯	1000
P&L on position 1	37	35	⋯	-32	⋯	-31	⋯	28	⋯	-12
P&L on position 2	-12	39	⋯	-10	⋯	31	⋯	23	⋯	-34
⋮	⋮	⋮		⋮		⋮		⋮		⋮
P&L on position N	60	-57	⋯	62	⋯	-54	⋯	53	⋯	-110
Total P&L	1250	1200	⋯	-850	⋯	-865	⋯	-875	⋯	-1100

6.4 Expected Shortfall

Up to this point, we have concentrated on VaR and related measures of risk (Marginal VaR and IVaR). Although VaR is the most widely used statistic in the marketplace, it has a few shortcomings. The most criticized drawback is that VaR is not a subadditive measure of risk. Subadditivity means that the risk of the sum of subportfolios is smaller than the sum of their individual risks. Since this property is not satisfied by VaR, it does not qualify as a "coherent" measure of risk as defined by Artzner, Delbaen, Eber and Heath (1999). Another criticism of VaR is that it does not provide an estimate for the size of losses in those scenarios where the VaR threshold is indeed exceeded. For example, if 95% VaR is USD 100, then we know that 5% of the time we will experience losses greater than USD 100, but we have no indication of how large those losses would be.

Expected Shortfall is a subadditive risk statistic that describes how large losses are on average when they exceed the VaR level, and hence it provides further information about the tail of the P&L distribution.[5] Mathematically, we can define Expected Shortfall as the conditional expectation of the portfolio losses given that they are greater than VaR. That is

$$\text{Expected Shortfall} = \mathbf{E}[-\Delta V \mid -\Delta V > \text{VaR}]. \tag{6.18}$$

While Expected Shortfall is currently not as widely used as VaR, it is a useful statistic that provides valuable additional information. In particular, Expected Shortfall allows the direct comparison of the tails of two

[5]This statistic is also called conditional VaR, mean excess loss, or tail VaR.

distributions. We can think of Expected Shortfall as an average measure of how heavy the tail of the distribution is.

Expected Shortfall also has some desirable mathematical properties that VaR lacks. For example, under some technical conditions, Expected Shortfall is a convex function of portfolio weights, which makes it extremely useful in solving optimization problems when we want to minimize the risk subject to certain constraints.[6]

Expected Shortfall and VaR can be combined to provide a measure of the cost of insuring portfolio losses beyond VaR at an α confidence level. That is,

$$\begin{aligned}
\text{Cost of Insurance} &= \mathbf{E}[\max(\mathrm{L} - \mathrm{VaR}, 0)] && (6.19)\\
&= \mathbf{E}[\max(\mathrm{L}, \mathrm{VaR})] - \mathrm{VaR} && (6.20)\\
&= \mathbf{E}[\mathrm{L} \times \mathbf{1}_{\mathrm{L}>\mathrm{VaR}} + \mathrm{VaR} \times \mathbf{1}_{\mathrm{L}\leq\mathrm{VaR}}] - \mathrm{VaR} && (6.21)\\
&= (1-\alpha)(\mathbf{E}[\mathrm{L}|\mathrm{L} > \mathrm{VaR}] - \mathrm{VaR}), && (6.22)
\end{aligned}$$

where $\mathrm{L} = -\Delta V$ are the portfolio losses.

The cost to insure losses is then interpreted as the expected loss minus VaR (because the contract pays for excess losses beyond VaR) multiplied by the probability that losses are larger than VaR (because insurance will pay only on that scenario).

We have said that Expected Shortfall has some desirable mathematical properties that make it a “well behaved” risk measure. In the next section, we will introduce a “coherent” family of risk measures that share a specific set of desirable properties and explain some of the inherent benefits of those properties.

6.5 Coherent risk measures

Artzner et al. (1999) define a family of risk measures that satisfy the following four properties:

1. Translation invariance. Adding cash to the portfolio decreases its risk by the same amount.[7]
2. Subadditivity. The risk of the sum of subportfolios is smaller or equal than the sum of their individual risks.
3. Positive homogeneity of degree 1. If we double the size of every position in a portfolio, the risk of the portfolio will be twice as large.
4. Monotonocity. If losses in portfolio A are larger than losses in portfolio B for all possible risk factor return scenarios, then the risk of portfolio A is higher than the risk of portfolio B.

[6] See Rockafellar and Uryasev (2000).

[7] This property is intuitive only when we measure risk in terms of the final net worth and not in terms of changes in value.

The subadditivity and homogeneity properties have interesting consequences in practical applications that deserve some attention. Subadditivity is required in connection with aggregation of risks across desks, business units, accounts, or subsidiary companies. The idea in general is that the risk of the total should not be larger than the sum of the risks of the parts. This is important when different business units calculate their risks independently and we want to get an idea of the total risk involved. If the risk measure is subadditive, we are always guaranteed to have a conservative estimate of the total risk. Subadditivity could also be a matter of concern for regulators, where firms might be motivated to break up into affiliates to satisfy capital requirements.

While VaR does not always satisfy the subadditivity property, it is difficult to come up in practice with cases where it is not satisfied. In addition, if our purpose is not to aggregate risks computed by independent units, but rather to allocate risk, we can use the Incremental VaR measure described in Section 6.3.

In Section 6.3, we showed that homogeneity is the only property required for the construction of an Incremental VaR measure. Similarly, we can use any homogeneous risk measure to define an incremental risk measure and extend the notion of Incremental VaR to the general notion of incremental risk, where risk is defined in terms of some homogeneous risk measure:

$$\text{IRISK}_i = w_i \frac{\partial RISK}{\partial w_i}. \tag{6.23}$$

We finalize this section by giving two examples of coherent risk measures. Our first example is the Expected Shortfall presented in the previous section, which is arguably the most popular coherent risk measure. Our second example of coherent risk measures is the maximum loss introduced by Studer (1999). The idea behind maximum loss—as its name indicates—is to identify the maximum loss or worst possible outcome from a set of scenarios called the "trust region". The problem can be cast in mathematical terms as

$$\begin{aligned} \text{Maximum Loss} = \quad & \min & \Delta V(\mathbf{r}) \\ & \text{subject to} & \mathbf{r} \epsilon A, \end{aligned} \tag{6.24}$$

where the trust region A is a set with probability α that contains the return scenario $\mathbf{r} = 0$.

It can be shown that if the P&L function (ΔV) is continuous, maximum loss as defined above is a more conservative risk measure than VaR.[8] The main shortcoming of maximum loss is that it is difficult to estimate for general P&L functions. If we approximate the P&L by a quadratic function, we can efficiently solve for the maximum loss using standard nonlinear optimization methods.

6.6 Benchmarking and relative statistics

Asset managers are particularly concerned with the risk that their portfolio will underperform a benchmark. In this case, the appropriate risk statistic is not the VaR of their portfolio, but rather the VaR of the deviations

[8] Some other technical conditions are required for this result. See Studer (1999).

of their portfolio with respect to the benchmark. We can extend this concept beyond VaR, and apply the same idea to obtain any statistic of the difference between the returns of a portfolio and its benchmark. Since these statistics are relative to a benchmark, in the rest of this section we will refer to them by their name preceded by the word "relative" (e.g., Relative VaR and Relative Expected Shortfall).

Risk managers use relative statistics to measure the potential underperformance of their portfolio against their customized benchmark. By potential underperformance, we mean the forecast of worst case underperformance over a given horizon. By monitoring Relative VaR, we can determine if the portfolio is within the risk tolerance level as prescribed by a mandate or an internal risk policy.

We can calculate relative statistics using the same methods described in this document, but applied to a new portfolio created by going long the reference portfolio and shorting each position on the benchmark. In order to make a relevant comparison of returns between the benchmark and the portfolio, we sometimes need to scale the benchmark so that its present value equals the present value of the position. In order to scale the benchmark, we simply multiply the number of units held in each position in the benchmark by the following amount:

$$\text{Multiplier} = \frac{\text{Present value of the portfolio}}{\text{Present value of the benchmark}}. \tag{6.25}$$

Example 6.1 Relative VaR on a fixed income portfolio

Our example portfolio consists of government securities across five countries: Canada, Germany, the United Kingdom, Japan, and the United States. These bonds are split in two groups: short-dated maturities ranging from one to three years and long-dated maturities ranging from seven to ten years.

As a first step, the customized benchmark is defined. Investments are made in five countries from the JP Morgan Global Bond Index according to a specific weighting plan and maturity range filtering. Table 6.2 describes the customized benchmark and presents the Relative VaR results.

In Table 6.2, we can see that the portfolio could potentially underperform the custom benchmark by 13 basis points with a probability of 95%.[9] At the country level, we see that the most conservative sectors of the portfolio are Canada, Germany, and the U.S. (2 basis points) and the most aggressive sector is Japan (11 basis points). In addition, the longer maturity portfolios taken together could potentially underperform the benchmark to a greater extent (10 basis points) than the shorter dated maturity portfolios (5 basis points). ■

An interesting property of Relative VaR is that it does not depend on the base currency. For example, if the Relative VaR is USD 100 in USD terms, and the current exchange rate is JPY 100 per USD, then the Relative VaR in JPY terms is JPY 10,000. In order to understand this property, we can think of VaR as an special case of Relative VaR, where the benchmark is a cash position in the base currency with a value equal

[9] The Relative VaR numbers are calculated at the 95% confidence level.

Table 6.2: Benchmark components and Relative VaR

	Weights		Relative VaR (bp)		
	1-3 yrs	7-10 yrs	1-3 yrs	7-10 yrs	Total
Canada	5%	8%	1	1	2
Germany	6%	4%	1	1	2
Japan	10%	8.50%	1	10	11
United Kingdom	9%	2.50%	5	0	5
United States	28%	19%	2	2	2
Total			5	10	13

to the present value of the portfolio. Therefore, in the same way that VaR changes when we change the base currency, Relative VaR only changes when we change the benchmark irrespective of the currency in which both the benchmark and the portfolio are represented.

Tracking error vs Relative VaR

Tracking error is a relative risk measure commonly used by asset managers. It is defined as the standard deviation of the excess returns (the difference between the portfolio returns and the benchmark returns), while Relative VaR is defined as a percentile of the distribution of the excess returns. In the case that the excess returns are normally distributed, the tracking error is equivalent to the Relative VaR with an 84% confidence level.[10]

Since tracking error does not differentiate between positive and negative excess returns, it has a natural interpretation only when the distribution of returns is symmetric. A portfolio (or a benchmark) containing options will have a skewed distribution of excess returns, and hence the tracking error might not accurately reflect the risk that returns on the portfolio could be significantly lower than the benchmark returns without any corresponding upside. Standard deviation alone would not reveal this problem. This means that tracking error represents only a subset of the information that can be extracted from Relative VaR.

In spite of the similarities between Relative VaR and tracking error, there is one fundamental difference: the tracking error takes the historical differences of the actual portfolio with respect to the benchmark. In other words, the change in the composition of the portfolio through time is taken into account. This means that the tracking error is a good measure of the historical performance of the manager. However, tracking error is not necessarily a good predictor of future excess returns. Relative VaR accomplishes this goal by forecasting the deviations of the current portfolio from the benchmark using the historical returns for the assets (equity, foreign exchange rates, interest rates).

[10] For a normal distribution the 0.8413 percentile corresponds to one standard deviation.

For example, rising tracking errors over time could indicate either a drift towards a more aggressive portfolio or a potential style change on the part of the portfolio manager. However, if a portfolio manager gradually changes his style by taking new bets on the market, it would take some time to capture this information in the corresponding tracking error. If we use Relative VaR instead of tracking error, we could monitor and assess the risk of the portfolio as it is being changed. VaR measures can be monitored on a daily basis to help identify a risk pattern for each portfolio manager. Monitoring of VaR over time allows for a timely recognition of style changes.

This concludes our discussion of risk statistics. In the next chapter, we will present different ways of constructing reports and presenting the information obtained through the risk statistics.

Chapter 7

Reports

The main goal of risk reports is to facilitate the clear and timely communication of risk exposures from the risk takers to senior management, shareholders, and regulators. Risk reports must summarize the risk characteristics of a portfolio, as well as highlight risk concentrations.[1] The objective of this chapter is to give an overview of the basic ways in which we can visualize and report the risk characteristics of a portfolio using the statistics described in Chapter 6. We will show how to study the risk attributes of a portfolio through its distribution. We will also explain how to identify the existence of risk concentrations in specific groups of positions. Finally, we illustrate ways to investigate the effect of various risk factors on the overall risk of the portfolio.

7.1 An overview of risk reporting

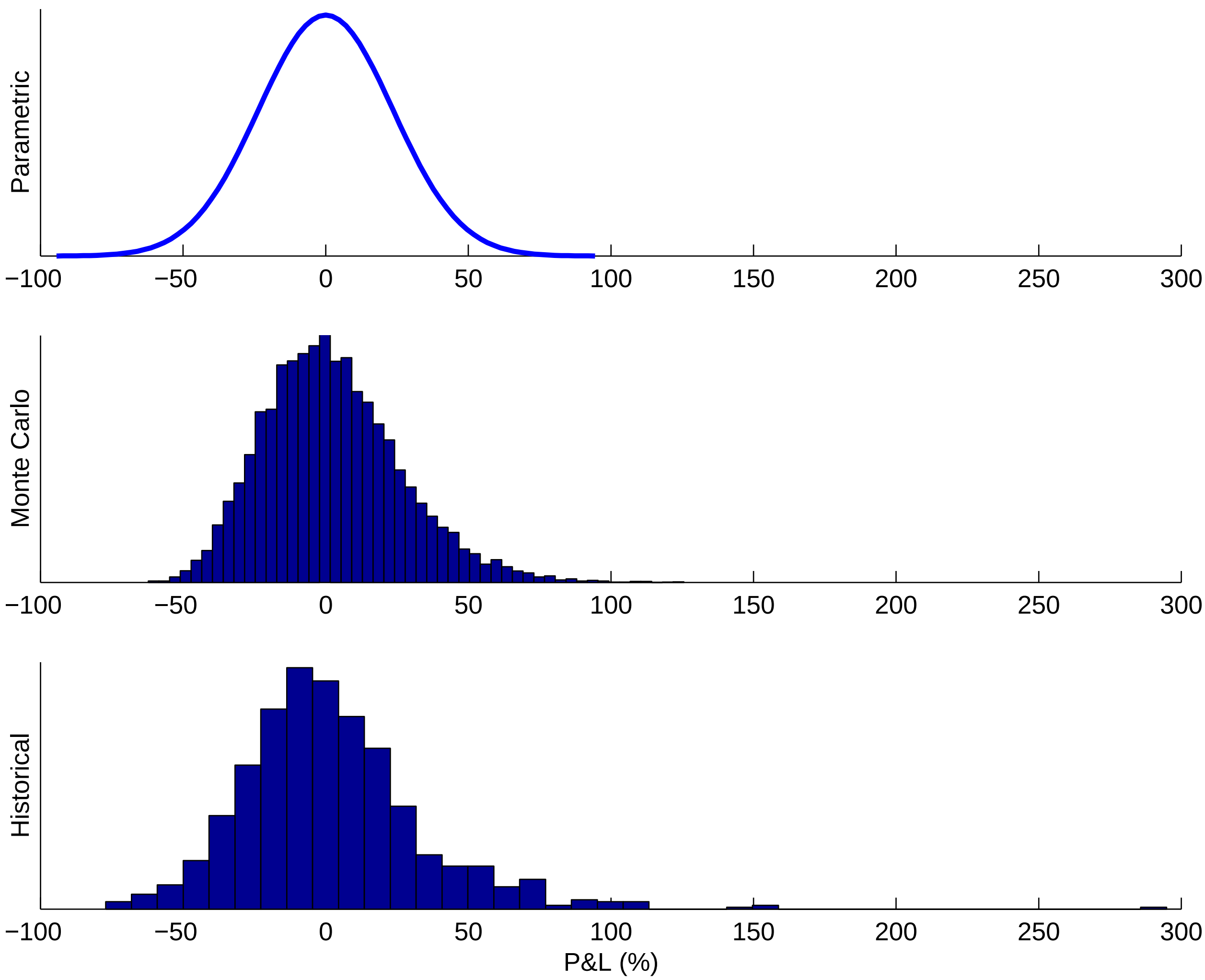
At the most aggregate level, we can depict in a histogram the entire distribution of future P&L values for our portfolio. We can construct a histogram using any of the methods described in Chapters 2 and 3 (i.e., Monte Carlo simulation, parametric, and historical simulation). The resulting distribution will depend on the assumptions made for each method. Figure 7.1 shows the histograms under each method for a one sigma out-of-the-money call option on the S&P 500.[2] Note that the parametric distribution is symmetric, while the Monte Carlo and historical distributions are skewed to the right. Moreover, the historical distribution assigns positive probability to high return scenarios not likely to be observed under the normality assumption for risk factor returns.

At a lower level of aggregation, we can use any of the risk measures described in Chapter 6 to describe particular features of the P&L distribution in more detail. For example, we can calculate the 95% VaR and

[1] For a detailed explanation of risk reporting practices see Laubsch (1999).

[2] Since the parametric distribution is normal, it can be drawn directly without the use of a histogram.

Figure 7.1: **Histogram reports**



expected shortfall from the distributions in Figure 7.1. Table 7.1 shows the results. Note that historical simulation provides the most conservative results, while Monte Carlo provides the lowest risk figures. The difference between the risk measures from these two methods is entirely due to the fact that risk factors have historically presented larger deviations than those implied by a normal distribution. This does not necessarily mean that historical simulation gives more accurate results since some of the historical returns could have been sampled from previous volatile periods that are no longer applicable to the current state of the world. We can also see from Table 7.1 that parametric VaR and expected shortfall are larger than the numbers obtained through Monte Carlo simulation, but smaller than the historical simulation figures. While parametric methods assume that instrument prices are linear functions of the risk factors, Monte Carlo and historical simulation do not make any assumptions regarding the pricing function (i.e., they use full pricing functions). In addition to the linearity assumption, parametric methods also assume a normal distribution of risk factor returns, which causes further discrepancies with the non-normal returns of historical simulation. In our example, the normality assumption causes parametric VaR to be smaller than historical VaR, while the linearity assumption removes the lower bounds imposed by the option payoff, causing parametric VaR to be higher than Monte Carlo VaR.

Table 7.1: 95% VaR and Expected Shortfall

	VaR	Expected Shortfall
Parametric	-39%	-49%
Monte Carlo	-34%	-40%
Historical	-42%	-53%

The comparison of results from different methods is useful to study the effect of our distributional assumptions, and estimate the potential magnitude of the error incurred by the use of a model. However, in practice, it is often necessary to select from the parametric, historical, and Monte Carlo methods to facilitate the flow of information and consistency of results throughout an organization.

The selection of the calculation method should depend on the specific portfolio and the choice of distribution of risk factor returns. If the portfolio consists mainly of linear positions and we choose to use a normal distribution of returns, then the best choice is the parametric method due to its speed and accuracy under those circumstances. If the portfolio consists mainly of non-linear positions, then we need to use either Monte Carlo or historical simulation depending on the desired distribution of returns. The selection between the normal and empirical distributions is usually done based on practical considerations rather than through statistical tests. The main reason to use the empirical distribution is to assign greater likelihood to large returns which have a small probability of occurrence under the normal distribution. However, there are a few drawbacks in the use of empirical distributions. The first problem is the difficulty of selecting the historical period used to construct the distribution. The risk estimates from historical simulation could present large differences based on the specific period chosen, and those differences could be difficult to explain when we have a large number of risk factors. In addition, the scarcity of historical data makes it difficult to extend the analysis horizon beyond a few days.[3]

Table 7.2 shows the best calculation method for each combination of portfolio characteristics and distribution. It is important to emphasize that Monte Carlo simulation also has the flexibility to incorporate non-normal distributions of risk factor returns.[4]

Table 7.2: Selecting a methodology

	Portfolio	
Distribution	Linear	Non-linear
Normal	Parametric	Monte Carlo
Non-normal	Historical	Historical

When dealing with complex or large portfolios, we will often need finer detail in the analysis. We can use risk measures to "dissect" risk across different dimensions and identify the sources of portfolio risk. This is useful to identify risk concentrations by business unit, asset class, country, currency, and maybe even all the way down to the trader or instrument level. For example, we can create a VaR table, where we show the risk of every business unit across rows, and counterparties across columns. Each entry on Table 7.3 represents the VaR of all the transactions made with a specific counterparty that we hold on a given portfolio. These different choices of rows and columns are called "drilldown dimensions". The total VaR of the portfolio is USD 1,645,689.

Table 7.3: VaR by business unit and counterparty

	Total	JPMorgan	Goldman Sachs	Morgan Stanley	Deutsche Bank
Fixed Income	279,521	555,187	279,521	90,818	
Foreign Exchange	132,662	58,165	90,818		
Global Equities	642,535			642,535	
Proprietary Trading	1,145,659	269,394	749,180	555,241	189,413
Total	1,645,689	259,300	957,381	435,910	189,413

The next section describes drilldowns in detail and explains how to calculate statistics in each of the buckets defined by a drilldown dimension.

[3]These issues are discussed in Chapter 3.
[4]In Appendix A we present a brief survey of the use of non-normal distributions in Monte Carlo methods.

7.2 Drilldowns

We refer to the categories in which you can slice the risk of a portfolio as "drilldown dimensions". Examples of drilldown dimensions are: position, portfolio, asset type, counterparty, currency, risk type (e.g., foreign exchange, interest rate, equity), and yield curve maturity buckets. We can divide drilldown dimensions in two broad groups. Proper dimensions are groups of positions. For all proper dimensions, a position is assigned to one, and only one, bucket. For example, a position corresponds to a single counterparty, a single portfolio, and a single asset type. We can create generic proper drilldown dimensions by assigning labels to each position. For example, we can create a label called "Region" having three values: Americas, Europe, and Asia, and use it as a drilldown dimension to report VaR by geographical region.

Improper drilldown dimensions are groups of risk factors. For an improper dimension, a position might correspond to more than one bucket. For example, an FX swap has both interest rate and FX risk. In addition, an FX swap has exposure to two different yield curves. Hence, risk type and yield curve are examples of improper dimensions.

Since proper drilldowns bucket positions into mutually exclusive sets, their calculation is rather straightforward: we simply construct subportfolios of positions for each drilldown dimension and calculate the corresponding statistic for each subportfolio. This process is the same whether we are using simulation or parametric methods. However, when we want to perform drilldowns on improper dimensions, simulation and parametric methods use different techniques.

7.2.1 Drilldowns using simulation methods

To produce a drilldown report for any statistic, we have to simulate changes in the risk factors contained in each bucket while keeping the remaining risk factors constant. In other words, for each scenario, we only use the changes in the risk factors that correspond to the dimension that we are analyzing. Once we have the change in value for each scenario on each bucket, we can calculate risk statistics using the ΔV information per bucket. In the following example, we illustrate the calculation of ΔV per bucket for one scenario.

Example 7.1 Drilldown by risk type and currency

In this example, we will calculate the change in value for a specific scenario in a portfolio consisting of a cash position of EUR one million, 13,000 shares of IBM, and a short position consisting of a one year at-the-money call on 20,000 shares of IBM with an implied volatility of 45.65%.[5]

[5] This is the portfolio of Example 3.1.

The current values and the new scenario for the risk factors are:

	Current Values	New Scenario
IBM	USD 120	USD 130
EUR	USD 0.88	USD 0.80
Discount Rate	6.0%	6.5%

Table 7.4 shows the original value of each instrument in the portfolio as well as their values under the new scenario. The last column shows the change in value (ΔV).

Table 7.4: Portfolio valuation under a new scenario

Position	Original Value (USD)	New Value (USD)	ΔV (USD)
Cash	880,000	800,000	-80,000
Equity	1,560,000	1,690,000	130,000
Option	-493,876	-634,472	-140,596
Total	1,946,123	1,855,527	-90,596

We can drilldown the value changes in Table 7.4 by risk type. To calculate the change in value due to equity changes, we would simply price the portfolio under a USD 130 IBM price scenario keeping the rest of the risk factors constant. This means that only the equity option and the equity will change. Similarly, for interest rate risk, we would price the option using a discount rate of 6.5%, but we will keep the price of IBM at its original value of USD 120. Table 7.5 shows ΔV drilled down by risk type. For example, if IBM gains USD 10, we gain USD 130,000 on our IBM position, but lose USD 134,581 on the option. This results on a loss of USD 4,581 in the portfolio due to a USD 10 increase in the price of IBM. Note that the total P&L of the portfolio is made up from the changes in value due to the equity, foreign exchange, and interest rate risk factors.

Table 7.5: Drilldown by risk type and position

	Total	Equity	Foreign Exchange	Interest Rate
Cash	-80,000		-80,000	
Equity	130,000	130,000		
Option	-140,596	-134,581		-5,227
Total	-90,596	-4,581	-80,000	-5,227

We can also drilldown ΔV by risk type and currency by selecting the risk factors that would change for each risk type/currency bucket. The risk factors that we would move to revalue the portfolio for each bucket are

	Total	Equity	Foreign Exchange	Interest Rate
USD	IBM and Discount Rate	IBM	None	Discount Rate
EUR	EUR/USD	None	EUR/USD	None
Total	All	IBM	EUR/USD	Discount Rate

Table 7.6 shows a ΔV drilldown report by risk type and currency. We can see that the total portfolio loss of USD 90,596 is made up of a loss of USD 10,596 due to changes in the USD denominated risk factors (IBM and discount rate), and a loss of USD 80,000 due to changes in the EUR denominated factors (EUR/USD exchange rate).

Table 7.6: ΔV drilldown by risk type and currency

	Total	Equity	Foreign Exchange	Interest Rate
USD	-10,596	-4,581		-5,227
EUR	-80,000		-80,000	
Total	-90,596	-4,581	-80,000	-5,227

■

Example 7.1 shows how to obtain a drilldown report for a ΔV scenario on two improper dimensions. It is easy to generalize this concept to any risk statistic by noting that all statistics are calculated from ΔV scenarios. Hence, we can obtain a drilldown report for any statistic by operating on ΔV buckets. In other words, we can think of Table 7.6 as giving us one ΔV scenario for each drilldown bucket. If we repeat the same exercise on different risk factor scenarios, we can create a drilldown report on any of the statistics described in Chapter 6. For example, let us say that we want to calculate VaR due to the fluctuations on USD denominated risk factors. From Table 7.6, we can see that our first ΔV scenario on this bucket is USD -10,596. This ΔV scenario resulted from a USD 130 price scenario for IBM and a 6.5% scenario for the discount rate. By using a different scenario for IBM and the discount rate, we can arrive at another ΔV scenario for the USD/Total bucket in Table 7.6, and by repeating this procedure we can generate a set of ΔV scenarios for the bucket that can be used to calculate VaR as explained in Section 6.1.1.

7.2.2 Drilldowns using parametric methods

Drilldowns using the parametric approach are based on delta equivalents rather than scenarios, that is, to calculate a risk statistic for each bucket, we set the delta equivalents falling outside the bucket equal to zero, and proceed to calculate the statistic as usual. This procedure is best explained with an example.

Example 7.2 Using delta equivalents in VaR drilldowns

In this example, we will use parametric methods to calculate a VaR report by risk type and currency for the portfolio in Example 7.1. The risk factors for the portfolio are IBM, the EUR/USD exchange rate, and a one-year zero-coupon bond. Table 7.7 shows the delta equivalents for the portfolio by position and risk factor. The columns in Table 7.7 contain the delta equivalent vectors for each position, as well as the total for the portfolio, while the rows contain the delta equivalents with respect to the corresponding risk factor broken down by position. Note that the sum of the delta equivalent vectors of the individual positions is equal to the delta equivalent vector of the portfolio, as explained in Section 2.3.

Table 7.7: Delta equivalents for the portfolio

Risk Factors	Total	Cash	Equity	Option
IBM	22,956	0	1,560,000	-1,537,043
EUR	880,000	880,000	0	0
1Y Bond	1,043,167	0	0	1,043,167

Let us assume that the covariance matrix of risk factor returns is:

$$\Sigma = \begin{pmatrix} 92.13 & -1.90 & 0.02 \\ -1.90 & 55.80 & -0.23 \\ 0.02 & -0.23 & 0.09 \end{pmatrix} \times 10^{-6} \tag{7.1}$$

From (6.4), we can calculate the one-day 95% VaR of the portfolio as $1.64\sqrt{\delta^\top \Sigma \delta} = \text{USD } 10{,}768$, where

$$\delta = \begin{pmatrix} 22{,}956 \\ 880{,}000 \\ 1{,}043{,}167 \end{pmatrix}. \tag{7.2}$$

To calculate the equity risk component of VaR, we simply create the modified delta equivalents vector

$$\delta_{Eq} = \begin{pmatrix} 22{,}956 \\ 0 \\ 0 \end{pmatrix}, \tag{7.3}$$

and calculate VaR as $1.64\sqrt{T\delta_{Eq}^{\top}\Sigma\delta_{Eq}}$ = USD 362. This means that 95% of the days we would not lose more than USD 362 in our portfolio due to changes in IBM equity. Similarly, if we wanted to obtain VaR for the FX bucket, we would simply zero out the first and third entries of δ and apply (6.4). Table 7.8 shows the one-day 95% VaR drilled down by risk type and currency. The VaR for the USD portion (USD 625) can be interpreted as the minimum amount we can expect to lose 1 out of 20 days—on average—due to fluctuations in USD denominated risk factors. Note that the sum of the VaRs for the USD and EUR buckets is greater than the total VaR of the portfolio due to diversification benefit (625 + 10,814 > 10,768). ■

Table 7.8: VaR drilldown by risk type and currency

	Total	Equity	Foreign Exchange	Interest Rate
USD	625	362		506
EUR	10,814		10,814	
Total	10,768	362	10,814	506

Since every statistic is calculated using delta equivalents under the parametric approach, we can always use this method to report drilldowns for every dimension. For example, (6.13) allows us to compute IVaR for arbitrary drilldown groups. Let us suppose that risk factors fall into one of the groups defined by the drilldown. For example, if the drilldown is asset type by currency, all the risk factors corresponding to bonds denominated in EUR will fall in one group. Then, we can define the delta equivalent vector for each group as the vector of factors contained in the group (and zeros elsewhere) and apply (6.13) directly.[6] Table 7.9 shows the one-day 95% incremental VaR drilled down by asset type and currency. Using this information, we can identify the positions that most contribute to the risk of the portfolio. For example, we can see that the largest contributor to risk in our portfolio are commodities, while the group of convertible bonds in our portfolio is diversifying risk away. We can also see that the risk factors denominated in JPY account for USD 283,728 of the total risk of the portfolio. Note that in this report we have combined a proper dimension (asset type) with an improper one (currency).

Up to this point, we have presented some of the most common and effective ways of presenting the risk information as well as the methods to break down the aggregate risk in different dimensions. We have also emphasized the importance of looking at risk in many different ways in order to reveal potential exposures or concentrations to groups of risk factors. In the following section, we present a case study that provides a practical application of the reporting concepts we have introduced.

[6] We need to make sure that every risk factor corresponds to one and only one group.

Table 7.9: Incremental VaR drill down by currency and asset type

	Total	AUD	JPY	USD
Bond	1,016		-79	1,095
Bond Option	7,408	7,408		
Callable Bond	1,473			1,473
Cap	-6,165			-6,165
Collar	18			18
Commodity	1,567,757			1,567,757
Convertible Bond	-29,293			-29,293
Equity	490,454		283,806	206,647
Equity Option	-462			-462
Floor	25			25
FRA	8,703			8,703
FRN	3			3
FX Option	3,712	2,659		1,054
Zero Coupon Bond	1,959			1,959
Total	2,046,609	10,067	283,728	1,752,814

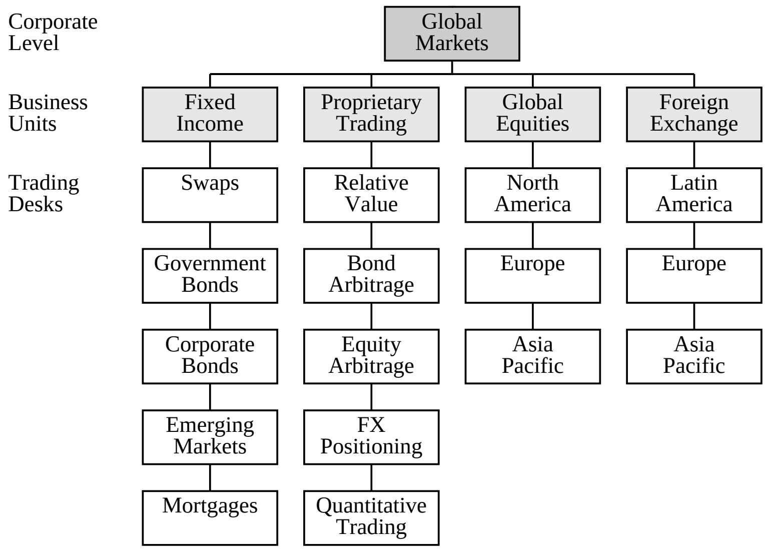
7.3 Global bank case study

Risk reporting is one of the most important aspects of risk management. Effective risk reports help understand the nature of market risks arising from different business units, countries, positions, and risk factors in order to prevent or act effectively in crisis situations. This section presents the example of a fictitious bank, ABC, which is structured in three organizational levels: corporate level, business units, and trading desks.[7] Figure 7.2 presents the organizational chart of ABC bank. The format and content of each risk report is designed to suit the needs of each organizational level.

At the corporate level, senior management needs a firmwide view of risk, and they will typically focus on market risk concentrations across business units as well as global stress test scenarios. Figure 7.3 shows the P&L distribution for the bank. Table 7.10 reports VaR by business unit and risk type. We can see that the one-day 95% VaR is USD 2,247,902. Among the business units, proprietary trading has the highest VaR level (USD 1,564,894), mainly as a result of their interest rate and equity exposures. However, the equity exposures in proprietary trading offset exposures in global equities resulting in a low total equity risk for the bank (USD 595,424). Similarly, the interest rate exposures taken by the proprietary trading unit are offsetting exposures in the emerging markets, fixed income, and foreign exchange units. We can also observe that the

[7] The case presented in Section 7.3 is based on Section 5.5 of Laubsch (1999).

Figure 7.2: **Organizational chart of ABC Bank**



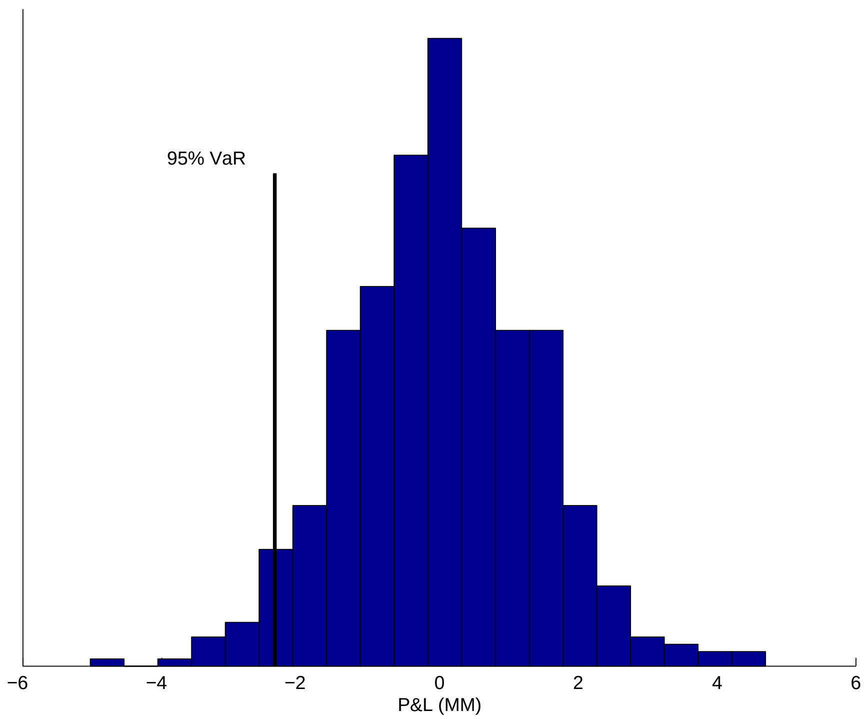
foreign exchange unit has a high interest rate risk reflecting the existence of FX forwards, futures, and options in their inventory.

Table 7.10: Corporate level VaR report

	Total	FX Risk	Interest Rate Risk	Equity Risk	Commodity Risk
Emerging Markets	758,248		758,348		
Fixed Income	393,131	80,765	381,807		
Foreign Exchange	181,207	124,052	79,449		
Global Equities	877,660			877,660	
Proprietary Trading	1,564,894	367,974	1,023,330	758,423	258,725
Total	2,247,902	396,756	1,307,719	595,424	258,725

Business units usually need to report risk by trading desk, showing more detail than the corporate reports. For example, a report at the business unit level might contain information by trading desk and country or currency. Table 7.11 reports VaR for the Foreign Exchange unit by trading desk and instrument type. For each instrument type the risk is reported by currency. We can see that most of ABC's FX risk is in cash (USD 148,102) with the single largest exposure denominated in JPY (USD 85,435). Also note that the FX

Figure 7.3: **ABC's P&L distribution**



Europe trading desk creates a concentration in EUR across cash, FX forward, and FX option instruments which accounts for most of its USD 93,456 at risk.

Table 7.11: Foreign exchange business unit VaR report

	Total	FX Latin America	FX Europe	FX Asia Pacific
Cash	148,102			
BRL		29,748		
EUR			51,820	
JPY				85,435
THB				8,453
FX Forward	61,568			
AUD				18,502
EUR			18,499	
GBP			11,781	
MXP		19,265		
FX Option	13,481			
EUR			18,535	
JPY				8,052
Total	181,207	41,456	93,456	109,678

At the trading desk level, risk information is presented at the most granular level. Trading desk reports might include detailed risk information by trader, position, and drilldowns such as yield curve positioning. Table 7.12 reports the VaR of the Government Bonds desk by trader and yield curve bucket. We can observe that Trader A is exposed only to fluctuations in the short end of the yield curve, while Trader C is well diversified across the entire term structure of interest rates. We can also see that trader B has a barbell exposure to interest rates in the intervals from six months to three years and fifteen years to thirty years. Note that the risk of the desk is diversified across the three traders. We can also see that the VaR of the Government Bonds desk (USD 122,522) is roughly one third of the VaR for the Fixed Income unit (USD 393,131).

Table 7.12: Government bonds desk VaR

	Total	0m-6m	6m-1y	1y-3y	3y-5y	5y-10y	10y-15y	15y-20y	20y-30y
Trader A	40,897	31,759	3,474	14,602					
Trader B	61,198	0	6,578	19,469				19,374	21,305
Trader C	26,876	6,104	1,187	5,673	1,229	6,598	9,515	1,127	516
Total	122,522	35,969	10,676	37,756	1,167	6,268	9,040	19,475	20,729

7.4 Conclusion

This document provides an overview of the methodology currently used by RiskMetrics in our market risk management applications. Part I discusses the mathematical assumptions of the multivariate normal model and the empirical model for the distribution of risk factor returns, as well as the parametric and simulation methods used to characterize such distributions. In addition, Part I gives a description of stress testing methods to complement the statistical models. Part II illustrates the different pricing approaches and the assumptions made in order to cover a wide set of asset types. Finally, Part III explains how to calculate risk statistics using the methods in Part I in conjunction with the pricing functions of Part II. We conclude the document by showing how to create effective risk reports based on risk statistics.

The models, assumptions, and techniques described in this document lay a solid methodological foundation for market risk measurement upon which future improvements will undoubtedly be made. We hope to have provided a coherent framework for understanding and using risk modeling techniques. As always, in the interest of improving and updating our methodologies, RiskMetrics appreciates and invites questions, comments, and criticisms.

Part IV

Appendices

Appendix A

Non-normal Distributions

A.1 The pros and cons of normal distributions

We have made extensive use of the normal distribution throughout this document. In fact, the models of Chapter 2 are based on a normal distribution of returns. In this appendix, we compare the normal distribution to some popular alternatives. We start by justifying the widespread practical use of normal distributions. It is important to keep in mind that the arguments we present are based on the specific problem at hand: the fast and accurate estimation of various risk statistics for a portfolio driven by a large number of risk factors. While the normal distribution has its shortcomings, its practical advantages in fitting individual asset returns and describing a dependence structure across many assets make it the best choice for our particular problem. This of course does not preclude the possibility of a better model for a specific problem, such as 99% VaR for an individual equity.

In Sections A.2 and A.3, we provide a brief survey of the most popular methods to model return distributions and discuss their strengths and weaknesses from a practical point of view.[1]

Since every distributional model has to consider the stand-alone characteristics of the returns as well as their dependence structure, we will divide the analysis in this section in two parts: univariate and multivariate.

A.1.1 Univariate normal distribution

The normal distribution has been widely used since the 18th century to model the relative frequency of physical and social phenomena. The use of normal distributions can be justified in theory as well as in

[1] Note that historical simulation also provides an alternative to the normal distribution. Since we have provided a detailed account of historical simulation in Chapter 3, we will not discuss it further in this appendix.

practice. The theoretical justification is given by the Central Limit Theorem (CLT). The CLT states that the distribution of a sum of a large number of independent and identically distributed variables is approximately normal. For example, we can model the price evolution of a risk factor as a series of successive shocks applied to the initial price. This implies that we can apply the CLT to logarithmic returns. A practical justification for the normal distribution is the simplicity of its calibration. The univariate normal distribution can be described by two parameters that are easy to calibrate: the mean and standard deviation. In Chapter 2, we discussed the calibration of these parameters.

In practice, empirical evidence shows that the unconditional normality assumption for logarithmic returns is not accurate. Many of the departures from normality can be explained by predictable changes in the dynamics of the return process; for instance, in many cases, the distribution of returns is closer to the normal distribution once time-varying volatility has been incorporated.[2] However, even after fitting sophisticated time series models to financial data, it is not uncommon to observe large events more frequently than predicted by the normal distribution. This phenomenon, typically referred to as heavy tails, has prompted academics and practitioners to investigate alternatives to the normal distribution. In the Section A.2, we provide a brief description of some of these alternatives.

A.1.2 Multivariate normal distribution

The most important practical advantage of the multivariate normal distribution is that its dependence structure is uniquely defined by a correlation matrix. This property is only shared by the elliptical family of distributions—those distributions for which the density function is constant on ellipsoids—which includes the multivariate normal and student-t distributions. Under most conditions, the multivariate normal distribution holds up well empirically. See for example Lopez and Walter (2000).

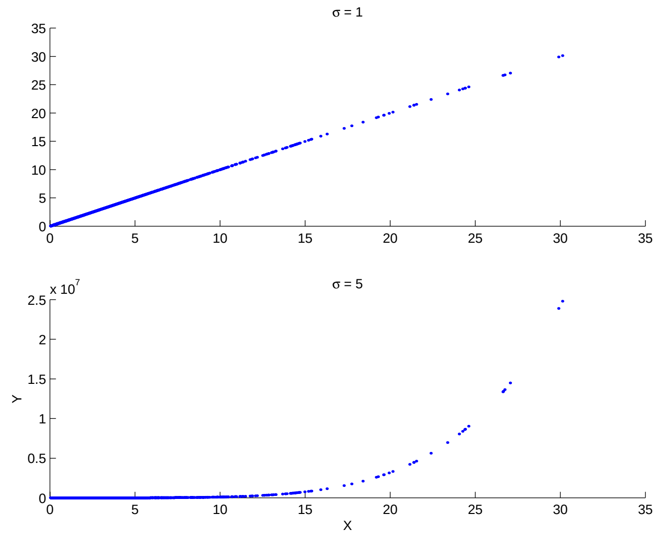
One shortcoming of the multivariate normal distribution is that even with strong correlations, large events seldom occur jointly for two assets. In fact, the distribution exhibits the property of asymptotic independence, which is discussed in detail in Embrechts, McNeil and Straumann (1999). In practice, one does observe large negative returns simultaneously for a variety of risk factors. To address this inadequacy of correlation measures under extreme market conditions, practitioners typically opt for historical simulations and stress testing.

A second shortcoming occurs when we consider univariate distributions outside the elliptical family. Here, the marginal (that is, univariate) distributions and the correlation matrix are no longer sufficient to describe the multivariate distribution of returns. Thus, correlation is not an adequate measure of dependence, meaning that even if two random variables have a high dependence on one another, their correlation coefficient could be low. For example, consider the case where Z is a standard normal random variable, and $X = \exp[Z]$ and $Y = \exp[\sigma Z]$ are two lognormal random variables.[3] Regardless of the value of σ, X and Y are perfectly

[2] See Morgan Guaranty Trust Company (1996).

[3] This example is taken from Embrechts, McNeil and Straumann (1999).

Figure A.1: **Correlation in the non-normal case**



dependent since one can be determined from the other with certainty. However, the correlation between X and Y depends on the value of σ. In Figure A.1.2, we present two scatter plots of X against Y. In the upper plot, for $\sigma = 1$, the correlation itself is perfect; in the lower plot, for $\sigma = 5$, the correlation coefficient, which is a measure only of linear dependence, is actually close to zero. The correlation coefficient is a deficient measure of dependence in this case since the true dependence between X and Y is non-linear. Outside of the elliptical family of distributions, this type of dependence is common, and thus the familiar notion of correlation can no longer serve as the sole dependence measure. Embrechts, McNeil and Straumann (1999) provide a detailed account of the uses and misuses of linear correlation in risk management and describe alternative dependence concepts.

A.2 Alternatives to the normal distribution

A.2.1 t distribution

The first alternative we will describe is the t distribution. The t distribution is an obvious first choice because it shares some of the attractive properties of the normal distribution while possessing heavy tails. Glasserman, Heidelberger and Shahabuddin (2000a) present a method to compute VaR using multivariate t distributions including some extensions using copula functions. See Lopez and Walter (2000) and references therein for further applications of the t distribution to VaR.

The univariate t distribution is described by only one parameter (the degrees of freedom) and is relatively easy to calibrate to market data. Koedjik, Huisman and Pownall (1998) suggest that the estimation of the degrees of freedom concentrate on the data in the tail of the distribution. The multivariate t distribution belongs to the elliptical family, meaning that it can be described by a correlation matrix.

One shortcoming of the multivariate t distribution is that all the marginal distributions must have the same degrees of freedom, which implies that every risk factor has equally heavy tails. The model can be extended to allow each risk factor to have different tails, however at the expense of abandoning the elliptical family. As mentioned before, outside the elliptical family there are many possible multivariate distributions with t marginal distributions. Hence, the cost of having different tails for each marginal is the additional task of specifying the multivariate distribution. In Section A.3 we present copula functions as a way of building a multivariate distribution from arbitrary desired marginal distributions.

A.2.2 Mixture distributions

Mixture distributions are used to model situations where the data can be viewed as arising from two or more distinct populations. In a risk management context, a mixture model is based on the observation that typically returns are moderate (quiet days), but from time to time are unusually large (hectic days). Under a mixture model, we specify the probability that a given day is quiet or hectic. We then specify that conditionally, returns are normally distributed, but with a low volatility on quiet days and a high volatility on hectic days. The resulting unconditional distribution, the mixture normal distribution, exhibits heavy tails due to the random nature of the volatility.

In order to fit a mixture normal distribution, we need to estimate five parameters: two means, two standard deviations, and the probability of having a hectic day. We can reduce the problem to the estimation of three parameters by setting both means to zero. The calibration in the multivariate case is more difficult, as we must estimate two covariance matrices corresponding to the quiet and hectic days, as well as the probability of a quiet day occurring. In general, the likelihood function for a mixture distribution has multiple maxima, making calibration difficult. In fact, there are special cases where the likelihood function is unbounded. For a reference on the calibration of mixture models see McLachlan and Basford (1987).

Practical applications of mixture models in risk management can be found in Zangari (1996), who uses a mixture normal to incorporate fat tails in the VaR calculation, and Finger and Kim (2000), who use a mixture model to assess correlation levels in stress situations.

A.2.3 Extreme Value Theory

Extreme Value Theory (EVT) provides a mathematical framework for the study of rare events. In the last few years, the potential application of EVT in risk management has received a fair amount of attention. In a nutshell, EVT provides the theory for describing extremes (maxima and minima) of random events.

In contrast to the CLT, which relates to the limit distribution of averages, EVT gives the limit distribution of maxima of independent, identically distributed random variables. For a detailed treatment of EVT, see Embrechts, Klüppelberg and Mikosch (1997).

EVT is supported by a solid theoretical ground, but its use has been limited due to practical considerations. The main advantage of EVT is that it can produce more accurate VaR estimates at high confidence levels. In addition, since EVT treats the upper and lower tails separately, it permits skewness of the return distribution. The practical limitations of EVT are that calibration is difficult and requires at least several years of daily observations. Further, the theory only allows for the treatment of low dimensional problems (e.g., portfolios with a very small number of risk factors). It is also important to note that EVT only provides information about extreme events and does not describe the middle part of the distribution.

See Embrechts (2000) for a discussion of the potential and limitations of EVT in risk management, and Embrechts, de Haan and Huang (1999) for a description of the multivariate problem.

To illustrate the differences between the normal, t, mixture normal, and generalized extreme value (GEV) distribution, we fit each of these distributions to daily returns from 1994 to 2000 for the JPY/USD exchange rate. Figure A.2.3 shows the cumulative distribution functions. We see that the largest deviations from the normal distribution are observed deep in the tails—at probability levels of 0.5% and smaller. It is important to note that the t and GEV distributions are calibrated only to the tails of the distribution and hence do not capture the behavior of more common returns. On the other hand, the mixture normal is fit to the entire distribution. In Figure A.2.3 we can observe that below the first percentile, the t, GED and mixture normal distributions are very similar. The mixture normal distribution suggests, however, that the phenomenon of heavy tails is only evident for returns beyond two standard deviations.

We have mentioned before that conditional volatility estimates can explain much of the non-normal behavior of asset returns. Thus, we also examine the JPY returns standardized by each day's RiskMetrics volatility forecast. The cumulative distribution functions for these conditional returns are displayed in Figure A.2.3. Deep into the tails, we can still observe large differences between the normal and other distributions. Nonetheless, the conditional distributions are closer than the unconditional distributions, particularly near the 95% VaR level.

A.3 Dependence and copulas

In Section A.1 we showed that outside the elliptical family of distributions, the linear correlation measure no longer provides an adequate indicator of dependence. Where linear correlation is lacking, there are several alternatives. One approach is to specify a more complex relationship between two random variables, such as the power law that links X and Y in our prior example. A second approach is to describe dependence through quantiles—that is, if X takes on a high value according to its marginal distribution, describe the likelihood that Y will take on a high value according to its marginal distribution. This class of measures, including rank

Figure A.2: **Unconditional cumulative distribution function for JPY returns**

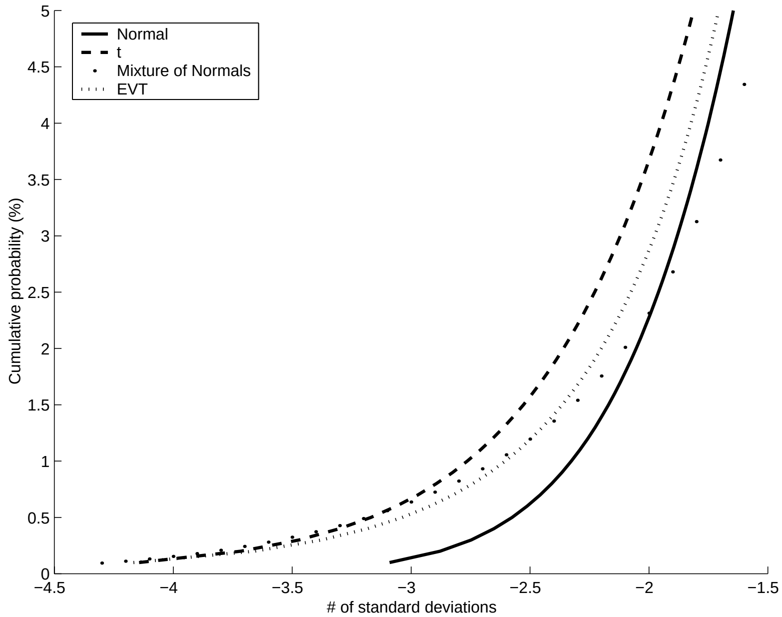
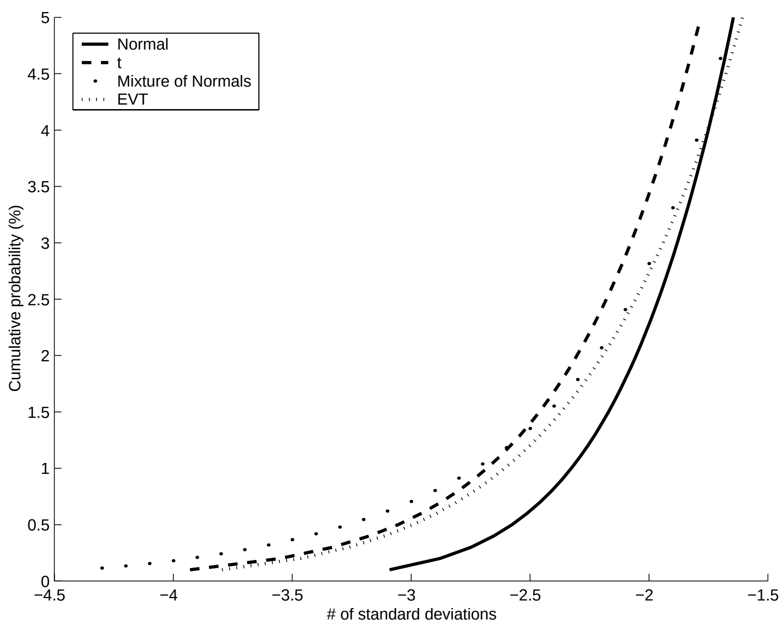


Figure A.3: **Conditional cumulative distribution function for JPY returns**



correlation measures are more robust than measures that stipulate a particular functional relationship (such as linear correlation); for instance, a measure of rank correlation describes the perfect dependence between the aforementioned X and Y. See Embrechts, McNeil and Straumann (1999) for further discussion of alternative dependence measures.

More concerning than how well a measure characterizes a dependence relationship is how to specify a multivariate distribution. While the measures we have discussed may be useful as descriptors, they only describe pairs of random variables. Outside the elliptical family, this is not sufficient information to specify the multivariate distribution. In general, for a set of random variables $X_1, \dots, X_n$, it is necessary to specify their joint distribution function

$$F(x_1, \dots, x_n) = \mathrm{P}[X_1 \leq x_1, \dots, X_n \leq x_n]. \tag{A.1}$$

For arbitrary marginal distributions, the problem of specifying F directly is intractable.

Copula functions provide a way to separate the joint distribution in two pieces: the dependence structure and the marginal return distributions. Let $F_1, \dots, F_n$ denote the marginal distributions for $X_1, \dots, X_n$. We can express the joint distribution function as

$$\begin{aligned} F(x_1, \dots, x_n) &= \mathrm{P}[F_1(X_1) \leq F_1(x_1), \dots, F_n(X_n) \leq F_n(x_n)] && \text{(A.2)} \\ &= C(F_1(x_1), \dots, F_n(x_n)). && \text{(A.3)} \end{aligned}$$

Here, we observe that the random variables $F_1(X_1), \dots, F_n(X_n)$ are each uniformly distributed. The function C—which is called the copula function—is a joint distribution function for a set of uniform random variables. Thus, given a set of random variables with known marginal distributions, the specification of their multivariate distribution reduces to the specification of the copula function C.

The copula framework fits naturally into Monte Carlo applications. For example, to generate realizations of $X_1, \dots, X_n$ using a normal copula, we proceed as follows:

1. Generate normal random variables $Z_1, \dots, Z_n$ with mean zero and correlation matrix Σ.

2. Transform the normal random variables to uniform random variables by applying the standard normal distribution function:

$$U_i = \Phi(Z_i), \text{ for } i = 1, \dots, n. \tag{A.4}$$

3. Transform the uniform random variables according to the desired marginal distribution functions:

$$X_i = F_i^{-1}(U_i), \text{ for } i = 1, \dots, n. \tag{A.5}$$

The first two steps produce correlated uniform random variables and accounts for the dependence structure. The third step accounts for the particular marginal distributions. Though we do not use it explicitly here, the copula function for this framework is

$$C(u_1, \ldots, u_n) = \Phi_n(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n); \Sigma), \tag{A.6}$$

where $\Phi_n(\cdot; \Sigma)$ is the distribution function for the multivariate normal distribution with mean zero and correlation matrix Σ.

Though extremely flexible, the copula approach does have practical limitations in a Monte Carlo setting. The generation of correlated uniform random variables, particularly with copula functions other than the normal, and the inversion of the distribution functions in (A.5) are both computationally taxing. For further references, see Frees and Valdez (1998) and Nelsen (1999).

Appendix B

RiskMetrics Publications

Throughout this document we make extensive references to various RiskMetrics publications. The purpose of this appendix is to provide a roadmap to RiskMetrics research publications in order to facilitate their use. Figure B shows the chronological evolution of our publications. All of our publications are available for download at `http://www.riskmetrics.com/research`

The seminal work was the first edition of the *RiskMetrics Technical Document* in 1994. In the next two years, we published another three editions of the Technical Document, the last of which is still available for download. The fourth edition of the Technical Document (*RiskMetrics Classic*) addresses the RiskMetrics methodology, data, and practical applications.

In 1995, a quarterly publication called RiskMetrics Monitor was created as a discussion forum to treat subjects of interest in market risk management. The Monitor reviewed issues affecting the users of RiskMetrics, from updates to the methodology, to reviews on how to evaluate market risk for specific instruments.

A parallel research effort in credit risk started in 1996, finally leading to the publication of the CreditMetrics Technical Document in April 1997. In 1998, our second periodical publication was created with an analogous purpose in the credit risk field. The CreditMetrics Monitor included the views of a wide range of contributors and encouraged open discussion of the methodology.

In 1999, we published three large documents. Our Practical Guide addresses the basic issues risk managers face when implementing a risk management process. The Guide is an extremely practical resource that discusses the methodology, the main reporting issues, and the analysis of the results. The CorporateMetrics Technical Document discusses the application of risk measurement techniques in non-financial corporations. In particular, CorporateMetrics is concerned with the potential impact of market rate fluctuations on a company's financial results. In the LongRun Technical Document, we present a framework to generate long-term market price and rate scenarios to measure risk over long horizons.

As a result of the increasing overlap in market and credit risk research, the RiskMetrics and CreditMetrics Monitors were merged in 2000. The new periodical was called RiskMetrics Journal. The aim of the Journal is to present advances in risk management and carry forward RiskMetrics' commitment to the transparency of our methodologies.

Here is a list of the RiskMetrics periodical publications from 1995 to 2000

RiskMetrics Journal

- November 2000
  - Calculating VaR through Quadratic Approximations
  - Hypothesis Test of Default Correlation and Application to Specific Risk
  - A Comparison of Stochastic Default Rate Models
- May 2000
  - Toward a Better Estimation of Wrong-Way Credit Exposure
  - Do Implied Volatilities Provide Early Warning for Market Stress?
  - A Stress Test to Incorporate Correlation Breakdown

CreditMetrics Monitor

- April 1999
  - Conditional Approaches for CreditMetrics Portfolio Distributions
  - The Valuation of Basket Credit Derivatives
  - An Analytic Approach for Credit Risk Analysis Under Correlated Defaults
- Third Quarter 1998
  - Extended "Constant Correlations" in CreditManager 2.0
  - Treating Collateral and Guarantees in CreditManager 2.0
  - Credit Derivatives in CreditMetrics
  - Commercial Paper Defaults and Rating Transitions, 1972 - 1998
  - A One-Parameter Representation of Credit Risk and Transition Matrices
- First Quarter 1998
  - Managing Credit Risk with CreditMetrics and Credit Derivatives
  - The Effect of Systematic Credit Risk on Loan Portfolio Value-at-Risk and Loan Pricing
  - Syndicated Bank Loan Recovery

  - Uses and Abuses of Bond Default Rates
  - Errata to the first edition of CreditMetrics Technical Document

RiskMetrics Monitor

- Special Edition 1998
  - How the Formation of the EMU Will Affect RiskMetrics
  - Overview of EMU, Resulting Changes in the RiskMetrics Methodology, and a Tool to Conduct Stress Testing on EMU-Related Scenarios
- Fourth Quarter 1997
  - A Methodology to Stress Correlations
  - What Risk Managers Should Know about Mean Reversion and Jumps in Prices
- Third Quarter 1997
  - An Investigation into Term Structure Estimation Methods for RiskMetrics
  - When is a Portfolio of Options Normally Distributed?
- Second Quarter 1997
  - A Detailed Analysis of a Simple Credit Exposure Calculator
  - A General Approach to Calculating VaR without Volatilities and Correlations
- First Quarter 1997
  - On Measuring Credit Exposure
  - The Effect of EMU on Risk Management
  - Streamlining the Market Risk Measurement Process
- Fourth Quarter 1996
  - Testing RiskMetrics Volatility Forecasts on Emerging Markets Data
  - When is Non-normality a Problem? The Case of 15 Time Series from Emerging Markets
- Third Quarter 1996
  - Accounting for Pull to Par and Roll Down for RiskMetrics Cashflows
  - How Accurate is the Delta-Gamma Methodology?

  - VaR for Basket Currencies

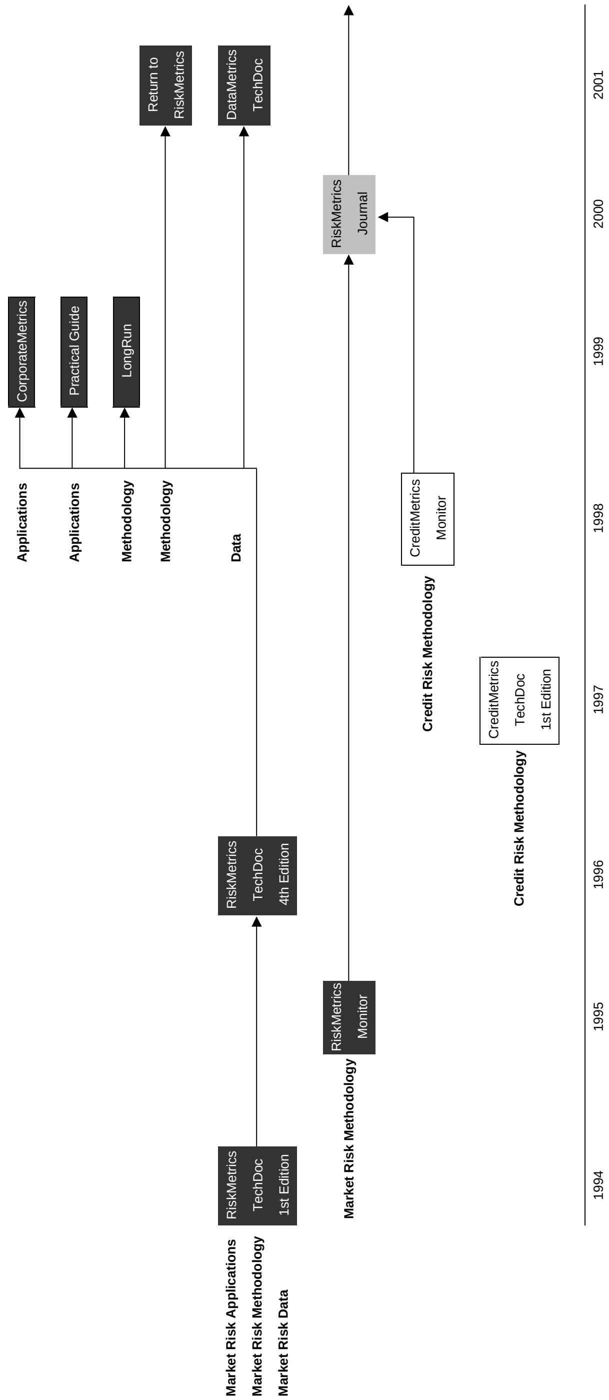
- Second Quarter 1996
  - An Improved Methodology for Measuring VaR that Allows for a More Realistic Model of Financial Return Tail Distributions
  - A Value-at-Risk Analysis of Currency Exposures
  - Underscoring the Limitations of Standard VaR When Underlying Market Return Distributions Deviate Significantly from Normality
  - Estimating Index Tracking Error for Equity Portfolios

- First Quarter 1996
  - A Look at Two Methodologies which Use a Basic Delta-Gamma Parametric VaR Precept but Achieve Similar Results to Simulation
  - Basel Committee Revises Market Risk Supplement to 1988 Capital Accord

- Fourth Quarter 1995
  - Exploring Alternative Volatility Forecasting Methods for the Standard RiskMetrics Monthly Horizon
  - How Accurate are the Risk Estimates in Portfolios which Contain Treasury Bills Proxied by LIBOR Data?
  - A Solution to the Standard Cash Flow Mapping Algorithm which Sometimes Leads to Imaginary Roots

- Third Quarter 1995
  - Mapping and Estimating VaR in Interest Rate Swaps
  - Adjusting Correlation from Nonsynchronous Data

Figure B.1: **Chronology of RiskMetrics publications**



Bibliography

Artzner, P., Delbaen, F., Eber, J. and Heath, D. (1999). Coherent Measures of Risk, *Mathematical Finance* **9**(3): 203–228.

Basel Committee on Banking Supervision (1996). Amendment to the Capital Accord to Incorporate Market Risks, Technical report, Bank for International Settlements. `http://www.bis.org`

Black, F. (1976). The Pricing of Commodity Contracts, *Journal of Financial Economics* **3**(1/2): 167–179.

Black, F. and Scholes, M. (1973). The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* **81**(3): 637–654.

Black, F., Derman, E. and Toy, W. (1990). A One-Factor Model of Interest Rates and its Application to Treasury Bond Options, *Financial Analysts Journal* **46**(1): 33–39.

Butler, J. and Schachter, B. (1998). Estimating Value-at-Risk with a Precision Measure by Combining Kernel Estimation with Historical Simulation, *Review of Derivatives Research* **1**: 371–390.

Curran, M. (1992). Beyond Average Intelligence, *Risk*.

Embrechts, P. (2000). Extreme Value Theory: Potential and Limitations as an Integrated Risk Management Tool, Working paper, ETH Zürich. `http://www.math.ethz.ch/~baltes/ftp/papers.html`

Embrechts, P., de Haan, L. and Huang, X. (1999). Modelling Multivariate Extremes, Working paper. `http://www.math.ethz.ch/~baltes/ftp/papers.html`

Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance*, Springer, Berlin.

Embrechts, P., McNeil, A. and Straumann, D. (1999). Correlation and Dependence in Risk Management: Properties and Pitfalls, Working paper, ETH Zürich. `http://www.risklab.ch/Papers.html`

Finger, C. C. and Kim, J. (2000). A Stress Test to Incorporate Correlation Breakdown, *Journal of Risk* **2**(3): 5–19.

Fleming, J., Kirby, C. and Ostdiek, B. (2001). The Economic Value of Volatility Timing, *Journal of Finance* **56**(1): 329–352.

Frees, E. W. and Valdez, E. A. (1998). Understanding Relationships Using Copulas, *North American Actuarial Journal* **2**(1): 1–20.

Glasserman, P., Heidelberger, P. and Shahabuddin, P. (2000a). Portfolio Value-at-Risk with Heavy Tailed Risk Factors, Working paper, Columbia Business School, PaineWebber working paper series. `http://www.columbia.edu/cu/business/wp`

Glasserman, P., Heidelberger, P. and Shahabuddin, P. (2000b). Variance Reduction Techniques for Estimating Value-at-Risk, *Management Science* **46**: 1349–1364.

Gupton, G. M., Finger, C. C. and Bhatia, M. (1997). *CreditMetrics – Technical Document*, Morgan Guaranty Trust Co. `http://www.riskmetrics.com/research/techdoc`

Hull, J. (1997). *Options, Futures, and Other Derivatives*, 4th edn, Princeton-Hall, Princeton, NJ.

Hull, J. and White, A. (1998). Incorporating Volatility Updating into the Historical Simulation Method for VaR, *Journal of Risk* **1**(1): 5–19.

Kim, J., Malz, A. M. and Mina, J. (1999). *LongRun Technical Document*, RiskMetrics Group.

Koedjik, K., Huisman, R. and Pownall, R. (1998). VaR-x: Fat tails in Financial Risk Management, *Journal of Risk* **1**(1): 47–61.

Kupiec, P. H. (1998). Stress Testing in a Value at Risk Framework, *Journal of Derivatives* **6**(1): 7–24.

Laubsch, A. J. (1999). *Risk Management: A Practical Guide*, RiskMetrics Group.

L'Ecuyer, P. (1988). *Communications of the ACM* **31**: 742–774.

Lopez, J. A. and Walter, C. A. (2000). Evaluating Covariance Matrix Forecasts in a Value-at-Risk Framework, *FRBSF working paper 2000-21*, Federal Reserve Bank of San Francisco.

Malz, A. (2000). Do Implied Volatilities Provide Early Warning of Market Stress?, *RiskMetrics Journal* **1**: 41–60.

Malz, A. (2001a). *DataMetrics Technical Document*, RiskMetrics Group, forthcoming.

Malz, A. (2001b). Vega Risk and the Smile, *Journal of Risk* **3**(2): 41–63.

McLachlan, G. J. and Basford, K. E. (1987). *Mixture Models: Inference and Applications to Clustering*, Marcel Dekker, New York and Basel.

Mina, J. (1999). Improved Cash-Flow Map, Working paper, RiskMetrics Group. `http://www.riskmetrics.com/research/working/cashflowmap.pdf`

Mina, J. (2000). Calculating VaR through Quadratic Approximations, *RiskMetrics Journal* **1**: 23–33.

Morgan Guaranty Trust Company (1996). *RiskMetrics Technical Document*, 4th edn.

Nelsen, R. (1999). *An Introduction to Copulas*, Springer, New York.

Press, W. H., Teukolsky, S. A., Vetterling, W. T. and Flannery, B. P. (1992). *Numerical Recipes in C: The Art of Scientific Computing*, 2nd edn, Cambridge University Press, Cambridge, U.K.

Rebonato, R. (1996). *Interest-Rate Option Models*, 1st edn, John Wiley & Sons.

Rockafellar, R. and Uryasev, S. (2000). Optimization of Conditional Value-at-Risk, *Journal of Risk* **2**(3): 21–41.

Studer, G. (1999). Market Risk Computation for Nonlinear Portfolios, *Journal of Risk* **1**(4): 33–53.

Trefethen, L.N. and Bau, D. III (1997). *Numerical Linear Algebra*, SIAM, Philadelphia, PA.

Turnbull, S. and Wakeman, L. (1991). A Quick Algorithm for Pricing European Average Options, *Journal of Financial and Quantitative Analysis* **26**(3): 377–389.

Zangari, P. (1996). An Improved Methodology for Measuring VaR, *RiskMetrics Monitor*, 2nd quarter pp. 7–25. `http://www.riskmetrics.com/research/journals`